IMO

INTERNATIONAL MATHEMATICAL OLYMPIAD

PROBLEMS AND SOLUTIONS

1959 - 2009

The most important and prestigious mathematical competition for high-school students

Contents

The International Mathematical Olympiad

History

The International Mathematical Olympiad (IMO) is the most important and prestigious mathematical competition for high-school students. It has played a significant role in generating wide interest in mathematics among high school students, as well as identifying talent.

In the beginning, the IMO was a much smaller competition than it is today. In 1959, the following seven countries gathered to compete in the first IMO: Bulgaria, Czechoslovakia, German Democratic Republic, Hungary, Poland, Romania, and the Soviet Union. Since then, the competition has been held annually. Gradually, other Eastern-block countries, countries from Western Europe, and ultimately numerous countries from around the world and every continent joined in. (The only year in which the IMO was not held was 1980, when for financial reasons no one stepped in to host it. Today this is hardly a problem, and hosts are lined up several years in advance.) In the 45th IMO, held in Athens, no fewer than 85 countries took part.

The Competition

The format of the competition quickly became stable and unchanging. Each country may send up to six contestants and each contestant competes individually (without any help or collaboration). The country also sends a team leader, who participates in problem selection and is thus isolated from the rest of the team until the end of the competition, and a deputy leader, who looks after the contestants.

The IMO competition lasts two days. On each day students are given four and a half hours to solve three problems, for a total of six problems. The first problem is usually the easiest on each day and the last problem the hardest, though there have been many notable exceptions. (IMO96-5 is one of the most difficult problems from all the Olympiads, having been fully solved by only six students out of several hundred!) Each problem is worth 7 points, making 42 points the maximum possible score. The number of points obtained by a contestant on each problem is the result of intense negotiations and, ultimately, agreement among the problem coordinators, assigned by the host country, and the team leader and deputy, who defend the interests of their contestants. This system ensures a relatively objective grade that is seldom off by more than two or three points.

Awards

Though countries naturally compare each other's scores, only individual prizes, namely medals and honorable mentions, are awarded on the IMO. Fewer than one twelfth of participants are awarded the gold medal, fewer than one fourth are awarded the gold or silver medal, and fewer than one half are awarded the gold, silver or bronze medal. Among the students not awarded a medal, those who score 7 points on at least one problem are awarded an honorable mention. This system of determining awards works rather well. It ensures, on the one hand, strict criteria and appropriate recognition for each level of performance, giving every contestant something to strive for. On the other hand, it also ensures a good degree of generosity that does not greatly depend on the variable difficulty of the problems proposed.

How the problems are selected

The selection of the problems consists of several steps. Participant countries send their proposals, which are supposed to be novel, to the IMO organizers. The organizing country does not propose problems. From the received proposals (the so-called longlisted problems), the Problem Committee selects a shorter list (the so-called shortlisted problems), which is presented to the IMO Jury, consisting of all the team leaders. From the short-listed problems the Jury chooses 6 problems for the IMO.

Apart from its mathematical and competitive side, the IMO is also a very large social event. After their work is done, the students have three days to enjoy the events and excursions organized by the host country, as well as to interact and socialize with IMO participants from around the world. All this makes for a truly memorable experience.

IMO Problems
1959 - 2009

First International Olympiad, 1959

1959/1.

Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number n.

1959/2.

For what real values of x is

$$\sqrt{(x+\sqrt{2x-1})}+\sqrt{(x-\sqrt{2x-1})}=A,$$

given (a) $A=\sqrt{2}$, (b) $A=1$, (c) $A=2$, where only non-negative real numbers are admitted for square roots?

1959/3.

Let a, b, c be real numbers. Consider the quadratic equation in $\cos x$:

$$a\cos^2 x+b\cos x+c=0.$$

Using the numbers a, b, c, form a quadratic equation in $\cos 2x$, whose roots are the same as those of the original equation. Compare the equations in $\cos x$ and $\cos 2x$ for $a=4, b=2, c=-1$.

1959/4.

Construct a right triangle with given hypotenuse c such that the median drawn to the hypotenuse is the geometric mean of the two legs of the triangle.

1959/5.

An arbitrary point M is selected in the interior of the segment AB. The squares $AMCD$ and $MBEF$ are constructed on the same side of AB, with the segments AM and MB as their respective bases. The circles circumscribed about these squares, with centers P and Q, intersect at M and also at another point N. Let N' denote the point of intersection of the straight lines AF and BC.
(a) Prove that the points N and N' coincide.
(b) Prove that the straight lines MN pass through a fixed point S independent of the choice of M.
(c) Find the locus of the midpoints of the segments PQ as M varies between A and B.

1959/6.

Two planes, P and Q, intersect along the line p. The point A is given in the plane P, and the point C in the plane Q; neither of these points lies on the straight line p. Construct an isosceles trapezoid $ABCD$ (with AB parallel to CD) in which a circle can be inscribed, and with vertices B and D lying in the planes P and Q respectively.

Second International Olympiad, 1960

1960/1.

Determine all three-digit numbers N having the property that N is divisible by 11, and $N/11$ is equal to the sum of the squares of the digits of N.

1960/2.

For what values of the variable x does the following inequality hold:

$$\frac{4x^2}{(1-\sqrt{1+2x})^2} < 2x + 9?$$

1960/3.

In a given right triangle ABC, the hypotenuse BC, of length a, is divided into n equal parts (n an odd integer). Let α be the acute angle subtending, from A, that segment which contains the midpoint of the hypotenuse. Let h be the length of the altitude to the hypotenuse of the triangle. Prove:

$$\tan \alpha = \frac{4nh}{(n^2-1)a}.$$

1960/4.

Construct triangle ABC, given h_a, h_b (the altitudes from A and B) and m_a, the median from vertex A.

1960/5.

Consider the cube $ABCDA'B'C'D'$ (with face $ABCD$ directly above face $A'B'C'D'$).

(a) Find the locus of the midpoints of segments XY, where X is any point of AC and Y is any point of $B'D'$.

(b) Find the locus of points Z which lie on the segments XY of part (a) with $ZY = 2XZ$.

1960/6.

Consider a cone of revolution with an inscribed sphere tangent to the base of the cone. A cylinder is circumscribed about this sphere so that one of its bases lies in the base of the cone. Let V_1 be the volume of the cone and V_2 the volume of the cylinder.

(a) Prove that $V_1 \neq V_2$.

(b) Find the smallest number k for which $V_1 = kV_2$, for this case, construct the angle subtended by a diameter of the base of the cone at the vertex of the cone.

1960/7.

An isosceles trapezoid with bases a and c and altitude h is given.
(a) On the axis of symmetry of this trapezoid, find all points P such that both legs of the trapezoid subtend right angles at P.
(b) Calculate the distance of P from either base.
(c) Determine under what conditions such points P actually exist. (Discuss various cases that might arise.)

Third International Olympiad, 1961

1961/1.

Solve the system of equations:

$$\begin{aligned} x+y+z &= a \\ x^2+y^2+z^2 &= b^2 \\ xy &= z^2 \end{aligned}$$

where a and b are constants. Give the conditions that a and b must satisfy so that x, y, z (the solutions of the system) are distinct positive numbers.

1961/2.

Let a, b, c be the sides of a triangle, and T its area. Prove: $a^2+b^2+c^2 \geq 4\sqrt{3}T$. In what case does equality hold?

1961/3.

Solve the equation $\cos^n x - \sin^n x = 1$, where n is a natural number.

1961/4.

Consider triangle $P_1P_2P_3$ and a point P within the triangle. Lines P_1P, P_2P, P_3P intersect the opposite sides in points Q_1, Q_2, Q_3 respectively. Prove that, of the numbers

$$\frac{P_1P}{PQ_1}, \frac{P_2P}{PQ_2}, \frac{P_3P}{PQ_3}$$

at least one is ≤ 2 and at least one is ≥ 2.

1961/5.

Construct triangle ABC if $AC = b, AB = c$ and $\angle AMB = \omega$, where M is the midpoint of segment BC and $\omega < 90°$. Prove that a solution exists if and only if

$$b \tan \frac{\omega}{2} \leq c < b.$$

In what case does the equality hold?

1961/6.

Consider a plane ε and three non-collinear points A, B, C on the same side of ε; suppose the plane determined by these three points is not parallel to ε. In plane a take three arbitrary points A', B', C'. Let L, M, N be the midpoints of segments AA', BB', CC'; let G be the centroid of triangle LMN. (We will not consider positions of the points A', B', C' such that the points L, M, N do not form a triangle.) What is the locus of point G as A', B', C' range independently over the plane ε?

Fourth International Olympiad, 1962

1962/1.

Find the smallest natural number n which has the following properties:
(a) Its decimal representation has 6 as the last digit.
(b) If the last digit 6 is erased and placed in front of the remaining digits, the resulting number is four times as large as the original number n.

1962/2.

Determine all real numbers x which satisfy the inequality:

$$\sqrt{3-x} - \sqrt{x+1} > \frac{1}{2}.$$

1962/3.

Consider the cube $ABCDA'B'C'D'$ ($ABCD$ and $A'B'C'D'$ are the upper and lower bases, respectively, and edges AA', BB', CC', DD' are parallel). The point X moves at constant speed along the perimeter of the square $ABCD$ in the direction $ABCDA$, and the point Y moves at the same rate along the perimeter of the square $B'C'CB$ in the direction $B'C'CBB'$. Points X and Y begin their motion at the same instant from the starting positions A and B', respectively. Determine and draw the locus of the midpoints of the segments XY.

1962/4.

Solve the equation $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$.

1962/5.

On the circle K there are given three distinct points A, B, C. Construct (using only straightedge and compasses) a fourth point D on K such that a circle can be inscribed in the quadrilateral thus obtained.

1962/6.

Consider an isosceles triangle. Let r be the radius of its circumscribed circle and ρ the radius of its inscribed circle. Prove that the distance d between the centers of these two circles is

$$d = \sqrt{r(r-2\rho)}.$$

1962/7.

The tetrahedron $SABC$ has the following property: there exist five spheres, each tangent to the edges $SA, SB, SC, BCCA, AB$, or to their extensions.
(a) Prove that the tetrahedron $SABC$ is regular.
(b) Prove conversely that for every regular tetrahedron five such spheres exist.

Fifth International Olympiad, 1963

1963/1.

Find all real roots of the equation

$$\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} = x,$$

where p is a real parameter.

1963/2.

Point A and segment BC are given. Determine the locus of points in space which are vertices of right angles with one side passing through A, and the other side intersecting the segment BC.

1963/3.

In an n-gon all of whose interior angles are equal, the lengths of consecutive sides satisfy the relation

$$a_1 \geq a_2 \geq \cdots \geq a_n.$$

Prove that $a_1 = a_2 = \cdots = a_n$.

1963/4.

Find all solutions x_1, x_2, x_3, x_4, x_5 of the system

$$\begin{aligned} x_5 + x_2 &= yx_1 \\ x_1 + x_3 &= yx_2 \\ x_2 + x_4 &= yx_3 \\ x_3 + x_5 &= yx_4 \\ x_4 + x_1 &= yx_5, \end{aligned}$$

where y is a parameter.

1963/5.

Prove that $\cos \frac{\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} = \frac{1}{2}$.

1963/6.

Five students, A, B, C, D, E, took part in a contest. One prediction was that the contestants would finish in the order $ABCDE$. This prediction was very poor. In fact no contestant finished in the position predicted, and no two contestants predicted to finish consecutively actually did so. A second prediction had the contestants finishing in the order $DAECB$. This prediction was better. Exactly two of the contestants finished in the places predicted, and two disjoint pairs of students predicted to finish consecutively actually did so. Determine the order in which the contestants finished.

Sixth International Olympiad, 1964

1964/1.

(a) Find all positive integers n for which $2^n - 1$ is divisible by 7.
(b) Prove that there is no positive integer n for which $2^n + 1$ is divisible by 7.

1964/2.

Suppose a, b, c are the sides of a triangle. Prove that

$$a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \leq 3abc.$$

1964/3.

A circle is inscribed in triangle ABC with sides a, b, c. Tangents to the circle parallel to the sides of the triangle are constructed. Each of these tangents cuts off a triangle from ΔABC. In each of these triangles, a circle is inscribed. Find the sum of the areas of all four inscribed circles (in terms of a, b, c).

1964/4.

Seventeen people correspond by mail with one another - each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.

1964/5.

Suppose five points in a plane are situated so that no two of the straight lines joining them are parallel, perpendicular, or coincident. From each point perpendiculars are drawn to all the lines joining the other four points. Determine the maximum number of intersections that these perpendiculars can have.

1964/6.

In tetrahedron $ABCD$, vertex D is connected with D_0 the centroid of ΔABC. Lines parallel to DD_0 are drawn through A, B and C. These lines intersect the planes BCD, CAD and ABD in points A_1, B_1 and C_1, respectively. Prove that the volume of $ABCD$ is one third the volume of $A_1B_1C_1D_0$. Is the result true if point D_0 is selected anywhere within ΔABC?

Seventh Internatioaal Olympiad, 1965

1965/1.

Determine all values x in the interval $0 \leq x \leq 2\pi$ which satisfy the inequality

$$2\cos x \leq \left|\sqrt{1+\sin 2x} - \sqrt{1-\sin 2x}\right| \leq \sqrt{2}.$$

1965/2.

Consider the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= 0 \end{aligned}$$

with unknowns x_1, x_2, x_3. The coefficients satisfy the conditions:
(a) a_{11}, a_{22}, a_{33} are positive numbers;
(b) the remaining coefficients are negative numbers;
(c) in each equation, the sum of the coefficients is positive.
Prove that the given system has only the solution $x_1 = x_2 = x_3 = 0$.

1965/3.

Given the tetrahedron $ABCD$ whose edges AB and CD have lengths a and b respectively. The distance between the skew lines AB and CD is d, and the angle between them is ω. Tetrahedron $ABCD$ is divided into two solids by plane ε, parallel to lines AB and CD. The ratio of the distances of ε from AB and CD is equal to k. Compute the ratio of the volumes of the two solids obtained.

1965/4.

Find all sets of four real numbers x_1, x_2, x_3, x_4 such that the sum of any one and the product of the other three is equal to 2.

1965/5.

Consider ΔOAB with acute angle AOB. Through a point $M \neq O$ perpendiculars are drawn to OA and OB, the feet of which are P and Q respectively. The point of intersection of the altitudes of ΔOPQ is H. What is the locus of H if M is permitted to range over (a) the side AB, (b) the interior of ΔOAB?

1965/6.

In a plane a set of n points ($n \geq 3$) is given. Each pair of points is connected by a segment. Let d be the length of the longest of these segments. We define a diameter of the set to be any connecting segment of length d. Prove that the number of diameters of the given set is at most n.

Eighth International Olympiad, 1966

1966/1.

In a mathematical contest, three problems, A, B, C were posed. Among the participants there were 25 students who solved at least one problem each. Of all the contestants who did not solve problem A, the number who solved B was twice the number who solved C. The number of students who solved only problem A was one more than the number of students who solved A and at least one other problem. Of all students who solved just one problem, half did not solve problem A. How many students solved only problem B?

1966/2.

Let a, b, c be the lengths of the sides of a triangle, and α, β, γ, respectively, the angles opposite these sides. Prove that if

$$a + b = \tan\frac{\gamma}{2}(a\tan\alpha + b\tan\beta),$$

the triangle is isosceles.

1966/3.

Prove: The sum of the distances of the vertices of a regular tetrahedron from the center of its circumscribed sphere is less than the sum of the distances of these vertices from any other point in space.

1966/4.

Prove that for every natural number n, and for every real number $x \neq k\pi/2^t (t = 0, 1, ..., n; k$ any integer)

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \cdots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x.$$

1966/5.

Solve the system of equations

$$\begin{array}{lllll}
 & |a_1 - a_2|\, x_2 & + |a_1 - a_3|\, x_3 & + |a_1 - a_4|\, x_4 & = 1 \\
|a_2 - a_1|\, x_1 & & + |a_2 - a_3|\, x_3 & + |a_2 - a_3|\, x_3 & = 1 \\
|a_3 - a_1|\, x_1 & + |a_3 - a_2|\, x_2 & & & = 1 \\
|a_4 - a_1|\, x_1 & + |a_4 - a_2|\, x_2 & + |a_4 - a_3|\, x_3 & & = 1
\end{array}$$

where a_1, a_2, a_3, a_4 are four different real numbers.

1966/6.

In the interior of sides BC, CA, AB of triangle ABC, any points K, L, M, respectively, are selected. Prove that the area of at least one of the triangles AML, BKM, CLK is less than or equal to one quarter of the area of triangle ABC.

Ninth International Olympiad, 1967

1967/1.

Let $ABCD$ be a parallelogram with side lengths $AB = a, AD = 1$, and with $\angle BAD = \alpha$. If ΔABD is acute, prove that the four circles of radius 1 with centers A, B, C, D cover the parallelogram if and only if

$$a \leq \cos\alpha + \sqrt{3}\sin\alpha.$$

1967/2.

Prove that if one and only one edge of a tetrahedron is greater than 1, then its volume is $\leq 1/8$.

1967/3.

Let k, m, n be natural numbers such that $m + k + 1$ is a prime greater than $n + 1$. Let $c_s = s(s + 1)$. Prove that the product

$$(c_{m+1} - c_k)(c_{m+2} - c_k)\cdots(c_{m+n} - c_k)$$

is divisible by the product $c_1c_2\cdots c_n$.

1967/4.

Let $A_0B_0C_0$ and $A_1B_1C_1$ be any two acute-angled triangles. Consider all triangles ABC that are similar to $\Delta A_1B_1C_1$ (so that vertices A_1, B_1, C_1 correspond to vertices A, B, C, respectively) and circumscribed about triangle $A_0B_0C_0$ (where A_0 lies on BC, B_0 on CA, and AC_0 on AB). Of all such possible triangles, determine the one with maximum area, and construct it.

1967/5.

Consider the sequence $\{c_n\}$, where

$$\begin{aligned}
c_1 &= a_1 + a_2 + \cdots + a_8 \\
c_2 &= a_1^2 + a_2^2 + \cdots + a_8^2 \\
&\cdots \\
c_n &= a_1^n + a_2^n + \cdots + a_8^n \\
&\cdots
\end{aligned}$$

in which $a_1, a_2, \cdots, a_8$ are real numbers not all equal to zero. Suppose that an infinite number of terms of the sequence $\{c_n\}$ are equal to zero. Find all natural numbers n for which $c_n = 0$.

1967/6.

In a sports contest, there were m medals awarded on n successive days ($n > 1$). On the first day, one medal and 1/7 of the remaining $m - 1$ medals were awarded. On the second day, two medals and 1/7 of the now remaining medals were awarded; and so on. On the n-th and last day, the remaining n medals were awarded. How many days did the contest last, and how many medals were awarded altogether?

Tenth International Olympiad, 1968

1968/1.

Prove that there is one and only one triangle whose side lengths are consecutive integers, and one of whose angles is twice as large as another.

1968/2.

Find all natural numbers x such that the product of their digits (in decimal notation) is equal to $x^2 - 10x - 22$.

1968/3.

Consider the system of equations

$$\begin{aligned} ax_1^2 + bx_1 + c &= x_2 \\ ax_2^2 + bx_2 + c &= x_3 \\ &\cdots \\ ax_{n-1}^2 + bx_{n-1} + c &= x_n \\ ax_n^2 + bx_n + c &= x_1, \end{aligned}$$

with unknowns $x_1, x_2, \cdots, x_n$, where a, b, c are real and $a \neq 0$. Let $\Delta = (b-1)^2 - 4ac$. Prove that for this system
(a) if $\Delta < 0$, there is no solution,
(b) if $\Delta = 0$, there is exactly one solution,
(c) if $\Delta > 0$, there is more than one solution.

1968/4.

Prove that in every tetrahedron there is a vertex such that the three edges meeting there have lengths which are the sides of a triangle.

1968/5.

Let f be a real-valued function defined for all real numbers x such that, for some positive constant a, the equation

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - [f(x)]^2}$$

holds for all x.
(a) Prove that the function f is periodic (i.e., there exists a positive number b such that $f(x+b) = f(x)$ for all x).
(b) For $a = 1$, give an example of a non-constant function with the required properties.

1968/6.

For every natural number n, evaluate the sum

$$\sum_{k=0}^{\infty}\left[\frac{n+2^k}{2^{k+1}}\right]=\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{4}\right]+\cdots+\left[\frac{n+2^k}{2^{k+1}}\right]+\cdots$$

(The symbol $[x]$ denotes the greatest integer not exceeding x.)

Eleventh International Olympiad, 1969

1969/1.

Prove that there are infinitely many natural numbers a with the following property: the number $z = n^4 + a$ is not prime for any natura1 number n.

1969/2.

Let $a_1, a_2, \cdots, a_n$ be real constants, x a real variable, and

$$\begin{aligned} f(x) &= \cos(a_1 + x) + \frac{1}{2}\cos(a_2 + x) + \frac{1}{4}\cos(a_3 + x) \\ &\quad + \cdots + \frac{1}{2^{n-1}}\cos(a_n + x). \end{aligned}$$

Given that $f(x_1) = f(x_2) = 0$, prove that $x_2 - x_1 = m\pi$ for some integer m.

1969/3.

For each value of $k = 1, 2, 3, 4, 5$, find necessary and sufficient conditions on the number $a > 0$ so that there exists a tetrahedron with k edges of length a, and the remaining $6 - k$ edges of length 1.

1969/4.

A semicircular arc γ is drawn on AB as diameter. C is a point on γ other than A and B, and D is the foot of the perpendicular from C to AB. We consider three circles, $\gamma_1, \gamma_2, \gamma_3$, all tangent to the line AB. Of these, γ_1 is inscribed in ΔABC, while γ_2 and γ_3 are both tangent to CD and to γ, one on each side of CD. Prove that γ_1, γ_2 and γ_3 have a second tangent in common.

1969/5.

Given $n > 4$ points in the plane such that no three are collinear. Prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals whose vertices are four of the given points.

1969/6.

Prove that for all real numbers $x_1, x_2, y_1, y_2, z_1, z_2$, with $x_1 > 0$, $x_2 > 0, x_1y_1 - z_1^2 > 0, x_2y_2 - z_2^2 > 0$, the inequality

$$\frac{8}{(x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2} \le \frac{1}{x_1y_1 - z_1^2} + \frac{1}{x_2y_2 - z_2^2}$$

is satisfied. Give necessary and sufficient conditions for equality.

Twelfth International Olympiad, 1970

1970/1.

Let M be a point on the side AB of ΔABC. Let r_1, r_2 and r be the radii of the inscribed circles of triangles AMC, BMC and ABC. Let q_1, q_2 and q be the radii of the escribed circles of the same triangles that lie in the angle ACB. Prove that

$$\frac{r_1}{q_1} \cdot \frac{r_2}{q_2} = \frac{r}{q}.$$

1970/2.

Let a, b and n be integers greater than 1, and let a and b be the bases of two number systems. A_{n-1} and A_n are numbers in the system with base a, and B_{n-1} and $B_n$are numbers in the system with base b; these are related as follows:

$$\begin{aligned} A_n &= x_n x_{n-1} \cdots x_0, A_{n-1} = x_{n-1} x_{n-2} \cdots x_0, \\ B_n &= x_n x_{n-1} \cdots x_0, B_{n-1} = x_{n-1} x_{n-2} \cdots x_0, \\ x_n &\neq 0, x_{n-1} \neq 0. \end{aligned}$$

Prove:

$$\frac{A_{n-1}}{A_n} < \frac{B_{n-1}}{B_n} \text{ if and only if } a > b.$$

1970/3.

The real numbers $a_0, a_1, ..., a_n, ...$ satisfy the condition:

$$1 = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots.$$

The numbers $b_1, b_2, ..., b_n, ...$ are defined by

$$b_n = \sum_{k=1}^{n} \left(1 - \frac{a_{k-1}}{a_k}\right) \frac{1}{\sqrt{a_k}}.$$

(a) Prove that $0 \leq b_n < 2$ for all n.
(b) Given c with $0 \leq c < 2$, prove that there exist numbers $a_0, a_1, ...$ with the above properties such that $b_n > c$ for large enough n.

1970/4.

Find the set of all positive integers n with the property that the set $\{n, n+1, n+2, n+3, n+4, n+5\}$ can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.

1970/5.

In the tetrahedron $ABCD$, angle BDC is a right angle. Suppose that the foot H of the perpendicular from D to the plane ABC is the intersection of the altitudes of ΔABC. Prove that

$$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2).$$

For what tetrahedra does equality hold?

1970/6.

In a plane there are 100 points, no three of which are collinear. Consider all possible triangles having these points as vertices. Prove that no more than 70% of these triangles are acute-angled.

Thirteenth International Olympiad, 1971

1971/1.

Prove that the following assertion is true for $n = 3$ and $n = 5$, and that it is false for every other natural number $n > 2$:
If $a_1, a_2, ..., a_n$ are arbitrary real numbers, then
$(a_1 - a_2)(a_1 - a_3) \cdots (a_1 - a_n) + (a_2 - a_1)(a_2 - a_3) \cdots (a_2 - a_n)$
$+ \cdots + (a_n - a_1)(a_n - a_2) \cdots (a_n - a_{n-1}) \geq 0$

1971/2.

Consider a convex polyhedron P_1 with nine vertices $A_1A_2, ..., A_9$; let P_i be the polyhedron obtained from P_1 by a translation that moves vertex A_1 to $A_i (i = 2, 3, ..., 9)$. Prove that at least two of the polyhedra $P_1, P_2, ..., P_9$ have an interior point in common.

1971/3.

Prove that the set of integers of the form $2^k - 3 (k = 2, 3, ...)$ contains an infinite subset in which every two members are relatively prime.

1971/4.

All the faces of tetrahedron $ABCD$ are acute-angled triangles. We consider all closed polygonal paths of the form $XYZTX$ defined as follows: X is a point on edge AB distinct from A and B; similarly, Y, Z, T are interior points of edges $BCCD, DA$, respectively. Prove:
(a) If $\angle DAB + \angle BCD \neq \angle CDA + \angle ABC$, then among the polygonal paths, there is none of minimal length.
(b) If $\angle DAB + \angle BCD = \angle CDA + \angle ABC$, then there are infinitely many shortest polygonal paths, their common length being $2AC \sin(\alpha/2)$, where $\alpha = \angle BAC + \angle CAD + \angle DAB$.

1971/5.

Prove that for every natural number m, there exists a finite set S of points in a plane with the following property: For every point A in S, there are exactly m points in S which are at unit distance from A.

1971/6.

Let $A = (a_{ij})(i, j = 1, 2, ..., n)$ be a square matrix whose elements are non-negative integers. Suppose that whenever an element $a_{ij} = 0$, the sum of the elements in the ith row and the jth column is $\geq n$. Prove that the sum of all the elements of the matrix is $\geq n^2/2$.

Fourteenth International Olympiad, 1972

1972/1.

Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint subsets whose members have the same sum.

1972/2.

Prove that if $n \geq 4$, every quadrilateral that can be inscribed in a circle can be dissected into n quadrilaterals each of which is inscribable in a circle.

1972/3.

Let m and n be arbitrary non-negative integers. Prove that

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer. $(0! = 1.)$

1972/4.

Find all solutions $(x_1, x_2, x_3, x_4, x_5)$ of the system of inequalities

$$\begin{aligned}
(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) &\leq 0 \\
(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) &\leq 0 \\
(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) &\leq 0 \\
(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) &\leq 0 \\
(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) &\leq 0
\end{aligned}$$

where x_1, x_2, x_3, x_4, x_5 are positive real numbers.

1972/5.

Let f and g be real-valued functions defined for all real values of x and y, and satisfying the equation

$$f(x+y) + f(x-y) = 2f(x)g(y)$$

for all x, y. Prove that if $f(x)$ is not identically zero, and if $|f(x)| \leq 1$ for all x, then $|g(y)| \leq 1$ for all y.

1972/6.

Given four distinct parallel planes, prove that there exists a regular tetrahedron with a vertex on each plane.

Fifteenth International Olympiad, 1973

1973/1.

Point O lies on line $g; \overrightarrow{OP_1}, \overrightarrow{OP_2}, ..., \overrightarrow{OP_n}$ are unit vectors such that points $P_1, P_2, ..., P_n$ all lie in a plane containing g and on one side of g. Prove that if n is odd,

$$\left|\overrightarrow{OP_1} + \overrightarrow{OP_2} + \cdots + \overrightarrow{OP_n}\right| \geq 1$$

Here $\left|\overrightarrow{OM}\right|$ denotes the length of vector $\overrightarrow{OM}$.

1973/2.

Determine whether or not there exists a finite set M of points in space not lying in the same plane such that, for any two points A and B of M, one can select two other points C and D of M so that lines AB and CD are parallel and not coincident.

1973/3.

Let a and b be real numbers for which the equation

$$x^4 + ax^3 + bx^2 + ax + 1 = 0$$

has at least one real solution. For all such pairs (a, b), find the minimum value of $a^2 + b^2$.

1973/4.

A soldier needs to check on the presence of mines in a region having the shape of an equilateral triangle. The radius of action of his detector is equal to half the altitude of the triangle. The soldier leaves from one vertex of the triangle. What path shouid he follow in order to travel the least possible distance and still accomplish his mission?

1973/5.

G is a set of non-constant functions of the real variable x of the form

$$f(x) = ax + b, a \text{ and } b \text{ are real numbers},$$

and G has the following properties:
(a) If f and g are in G, then $g \circ f$ is in G; here $(g \circ f)(x) = g[f(x)]$.
(b) If f is in G, then its inverse f^{-1} is in G; here the inverse of $f(x) = ax + b$ is $f^{-1}(x) = (x - b)/a$.
(c) For every f in G, there exists a real number x_f such that $f(x_f) = x_f$.
Prove that there exists a real number k such that $f(k) = k$ for all f in G.

1973/6.

Let $a_1, a_2, ..., a_n$ be n positive numbers, and let q be a given real number such that $0 < q < 1$. Find n numbers $b_1, b_2, ..., b_n$ for which

(a) $a_k < b_k$ for $k = 1, 2, \cdots, n$,

(b) $q < \frac{b_{k+1}}{b_k} < \frac{1}{q}$ for $k = 1, 2, ..., n-1$,

(c) $b_1 + b_2 + \cdots + b_n < \frac{1+q}{1-q}(a_1 + a_2 + \cdots + a_n)$.

Sixteenth International Olympiad, 1974

1974/1.

Three players A, B and C play the following game: On each of three cards an integer is written. These three numbers p, q, r satisfy $0 < p < q < r$. The three cards are shuffled and one is dealt to each player. Each then receives the number of counters indicated by the card he holds. Then the cards are shuffled again; the counters remain with the players.
This process (shuffling, dealing, giving out counters) takes place for at least two rounds. After the last round, A has 20 counters in all, B has 10 and C has 9. At the last round B received r counters. Who received q counters on the first round?

1974/2.

In the triangle ABC, prove that there is a point D on side AB such that CD is the geometric mean of AD and DB if and only if

$$\sin A \sin B \leq \sin^2 \frac{C}{2}.$$

1974/3.

Prove that the number $\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$ is not divisible by 5 for any integer $n \geq 0$.

1974/4.

Consider decompositions of an 8×8 chessboard into p non-overlapping rectangles subject to the following conditions:
(i) Each rectangle has as many white squares as black squares.
(ii) If a_i is the number of white squares in the i-th rectangle, then $a_1 < a_2 < \cdots < a_p$. Find the maximum value of p for which such a decomposition is possible. For this value of p, determine all possible sequences $a_1, a_2, \cdots, a_p$.

1974/5.

Determine all possible values of

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

where a, b, c, d are arbitrary positive numbers.

1974/6.

Let P be a non-constant polynomial with integer coefficients. If $n(P)$ is the number of distinct integers k such that $(P(k))^2 = 1$, prove that $n(P) - \deg(P) \leq 2$, where $\deg(P)$ denotes the degree of the polynomial P.

Seventeenth International Olympiad, 1975

1975/1.

Let x_i, y_i $(i = 1, 2, ..., n)$ be real numbers such that

$$x_1 \geq x_2 \geq \cdots \geq x_n \text{ and } y_1 \geq y_2 \geq \cdots \geq y_n.$$

Prove that, if $z_1, z_2, \cdots, z_n$ is any permutation of $y_1, y_2, \cdots, y_n$, then

$$\sum_{i=1}^{n}(x_i - y_i)^2 \leq \sum_{i=1}^{n}(x_i - z_i)^2.$$

1975/2.

Let $a_1, a_2, a_{3,} \cdots$ be an infinite increasing sequence of positive integers. Prove that for every $p \geq 1$ there are infinitely many a_m which can be written in the form

$$a_m = xa_p + ya_q$$

with x, y positive integers and $q > p$.

1975/3.

On the sides of an arbitrary triangle ABC, triangles ABR, BCP, CAQ are constructed externally with $\angle CBP = \angle CAQ = 45°, \angle BCP = \angle ACQ = 30°, \angle ABR = \angle BAR = 15°$. Prove that $\angle QRP = 90°$ and $QR = RP$.

1975/4.

When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B. (A and B are written in decimal notation.)

1975/5.

Determine, with proof, whether or not one can find 1975 points on the circumference of a circle with unit radius such that the distance between any two of them is a rational number.

1975/6.

Find all polynomials P, in two variables, with the following properties:
(i) for a positive integer n and all real t, x, y

$$P(tx, ty) = t^n P(x, y)$$

(that is, P is homogeneous of degree n),
(ii) for all real a, b, c,

$$P(b+c, a) + P(c+a, b) + P(a+b, c) = 0,$$

(iii) $P(1, 0) = 1$.

Eighteenth International Olympiad, 1976

1976/1.

In a plane convex quadrilateral of area 32, the sum of the lengths of two opposite sides and one diagonal is 16. Determine all possible lengths of the other diagonal.

1976/2.

Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for $j = 2, 3, \cdots$. Show that, for any positive integer n, the roots of the equation $P_n(x) = x$ are real and distinct.

1976/3.

A rectangular box can be filled completely with unit cubes. If one places as many cubes as possible, each with volume 2, in the box, so that their edges are parallel to the edges of the box, one can fill exactly 40% of the box. Determine the possible dimensions of all such boxes.

1976/4.

Determine, with proof, the largest number which is the product of positive integers whose sum is 1976.

1976/5.

Consider the system of p equations in $q = 2p$ unknowns $x_1, x_2, \cdots, x_q$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1q}x_q &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2q}x_q &= 0 \\ &\cdots \\ a_{p1}x_1 + a_{p2}x_2 + \cdots + a_{pq}x_q &= 0 \end{aligned}$$

with every coefficient a_{ij} member of the set $\{-1, 0, 1\}$. Prove that the system has a solution $(x_1, x_2, \cdots, x_q)$ such that

(a) all x_j $(j = 1, 2, ..., q)$ are integers,

(b) there is at least one value of j for which $x_j \neq 0$,

(c) $|x_j| \leq q(j = 1, 2, ..., q)$.

1976/6.

A sequence $\{u_n\}$ is defined by

$$u_0 = 2, u_1 = 5/2, u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1 \text{for } n = 1, 2, \cdots$$

Prove that for positive integers n,

$$[u_n] = 2^{[2^n-(-1)^n]/3}$$

where $[x]$ denotes the greatest integer $\leq x$.

Nineteenth International Mathematical Olympiad, 1977

1977/1.

Equilateral triangles ABK, BCL, CDM, DAN are constructed inside the square $ABCD$. Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments $AKBK, BL, CL, CM, DM, DN, AN$ are the twelve vertices of a regular dodecagon.

1977/2.

In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

1977/3.

Let n be a given integer > 2, and let V_n be the set of integers $1 + kn$, where $k = 1, 2, ...$. A number $m \in V_n$ is called *indecomposable* in V_n if there do not exist numbers $p, q \in V_n$ such that $pq = m$. Prove that there exists a number $r \in V_n$ that can be expressed as the product of elements indecomposable in V_n in more than one way. (Products which differ only in the order of their factors will be considered the same.)

1977/4.

Four real constants a, b, A, B are given, and

$$f(\theta) = 1 - a\cos\theta - b\sin\theta - A\cos 2\theta - B\sin 2\theta.$$

Prove that if $f(\theta) \geq 0$ for all real θ, then

$$a^2 + b^2 \leq 2 \text{ and } A^2 + B^2 \leq 1.$$

1977/5.

Let a and b be positive integers. When a^2+b^2 is divided by $a+b$, the quotient is q and the remainder is r. Find all pairs (a, b) such that $q^2 + r = 1977$.

1977/6.

Let $f(n)$ be a function defined on the set of all positive integers and having all its values in the same set. Prove that if

$$f(n+1) > f(f(n))$$

for each positive integer n, then

$$f(n) = n \text{ for each } n.$$

Twentieth International Olympiad, 1978

1978/1. m and n are natural numbers with $1 \leq m < n$. In their decimal representations, the last three digits of 1978^m are equal, respectively, to the last three digits of 1978^n. Find m and n such that $m + n$ has its least value.

1978/2. P is a given point inside a given sphere. Three mutually perpendicular rays from P intersect the sphere at points U, V, and $W; Q$ denotes the vertex diagonally opposite to P in the parallelepiped determined by PU, PV, and PW. Find the locus of Q for all such triads of rays from P

1978/3. The set of all positive integers is the union of two disjoint subsets $\{f(1), f(2), ..., f(n), ...\}, \{g(1), g(2), ..., g(n), ...\}$, where

$$f(1) < f(2) < \cdots < f(n) < \cdots,$$

$$g(1) < g(2) < \cdots < g(n) < \cdots,$$

and

$$g(n) = f(f(n)) + 1 \text{for all } n \geq 1.$$

Determine $f(240)$.

1978/4. In triangle $ABC, AB = AC$. A circle is tangent internally to the circumcircle of triangle ABC and also to sides AB, AC at P, Q, respectively. Prove that the midpoint of segment PQ is the center of the incircle of triangle ABC.

1978/5. Let $\{a_k\}(k = 1, 2, 3, ..., n, ...)$ be a sequence of distinct positive integers. Prove that for all natural numbers n,

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}.$$

1978/6. An international society has its members from six different countries. The list of members contains 1978 names, numbered $1, 2, ..., 1978$. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country.

Twenty-first International Olympiad, 1979

1979/1. Let p and q be natural numbers such that

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319}.$$

Prove that p is divisible by 1979.

1979/2. A prism with pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$ as top and bottom faces is given. Each side of the two pentagons and each of the line-segments A_iB_j for all $i, j = 1, ..., 5$, is colored either red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been colored has two sides of a different color. Show that all 10 sides of the top and bottom faces are the same color.

1979/3. Two circles in a plane intersect. Let A be one of the points of intersection. Starting simultaneously from A two points move with constant speeds, each point travelling along its own circle in the same sense. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that, at any time, the distances from P to the moving points are equal.

1979/4. Given a plane π, a point P in this plane and a point Q not in π, find all points R in π such that the ratio $(QP + PA)/QR$ is a maximum.

1979/5. Find all real numbers a for which there exist non-negative real numbers x_1, x_2, x_3, x_4, x_5 satisfying the relations

$$\sum_{k=1}^{5} kx_k = a, \sum_{k=1}^{5} k^3x_k = a^2, \sum_{k=1}^{5} k^5x_k = a^3.$$

1979/6. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there.. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that $a_{2n-1} = 0$,

$$a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1}), n = 1, 2, 3, \cdots,$$

where $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$.

Note. A path of n jumps is a sequence of vertices $(P_0, ..., P_n)$ such that
(i) $P_0 = A, P_n = E$;
(ii) for every $i, 0 \leq i \leq n-1, P_i$ is distinct from E;
(iii) for every $i, 0 \leq i \leq n-1, P_i$ and P_{i+1} are adjacent.

Twenty-second International Olympiad, 1981

1981/1. P is a point inside a given triangle $ABC.D, E, F$ are the feet of the perpendiculars from P to the lines BC, CA, AB respectively. Find all P for which

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$$

is least.

1981/2. Let $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1, 2, ..., n\}$. Each of these subsets has a smallest member. Let $F(n, r)$ denote the arithmetic mean of these smallest numbers; prove that

$$F(n, r) = \frac{n+1}{r+1}.$$

1981/3. Determine the maximum value of m^3+n^3,where m and n are integers satisfying $m, n \in \{1, 2, ..., 1981\}$ and $(n^2 - mn - m^2)^2 = 1$.

1981/4. (a) For which values of $n > 2$ is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining $n - 1$ numbers?

(b) For which values of $n > 2$ is there exactly one set having the stated property?

1981/5. Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle and the point O are collinear.

1981/6. The function $f(x, y)$ satisfies

(1) $f(0, y) = y + 1$,

(2)$f(x + 1, 0) = f(x, 1)$,

(3) $f(x + 1, y + 1) = f(x, f(x + 1, y))$,

for all non-negative integers x, y. Determine $f(4, 1981)$.

Twenty-third International Olympiad, 1982

1982/1. The function $f(n)$ is defined for all positive integers n and takes on non-negative integer values. Also, for all m, n

$$f(m+n) - f(m) - f(n) = 0 \text{ or } 1$$

$$f(2) = 0, f(3) > 0, \text{ and } f(9999) = 3333.$$

Determine $f(1982)$.

1982/2. A non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the side opposite A_i). For all $i = 1, 2, 3$, M_i is the midpoint of side a_i, and T_i. is the point where the incircle touches side a_i. Denote by S_i the reflection of T_i in the interior bisector of angle A_i. Prove that the lines M_1, S_1, M_2S_2, and M_3S_3 are concurrent.

1982/3. Consider the infinite sequences $\{x_n\}$ of positive real numbers with the following properties:

$$x_0 = 1, \text{and for all } i \geq 0, x_{i+1} \leq x_i.$$

(a) Prove that for every such sequence, there is an $n \geq 1$ such that

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

(b) Find such a sequence for which

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} < 4.$$

1982/4. Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers (x, y), then it has at least three such solutions. Show that the equation has no solutions in integers when $n = 2891$.

1982/5. The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by the inner points M and N, respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M, and N are collinear.

1982/6. Let S be a square with sides of length 100, and let L be a path within S which does not meet itself and which is composed of line segments $A_0A_1, A_1A_2, \cdots, A_{n-1}A_n$ with $A_0 \neq A_n$. Suppose that for every point P of the boundary of S there is a point of L at a distance from P not greater than $1/2$. Prove that there are two points X and Y in L such that the distance between X and Y is not greater than 1, and the length of that part of L which lies between X and Y is not smaller than 198.

Twenty-fourth International Olympiad, 1983

1983/1. Find all functions f defined on the set of positive real numbers which take positive real values and satisfy the conditions:
(i) $f(xf(y)) = yf(x)$ for all positive x, y;
(ii) $f(x) \to 0$ as $x \to \infty$.

1983/2. Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centers O_1 and O_2, respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1,and M_2 be the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.

1983/3. Let a, b and c be positive integers, no two of which have a common divisor greater than 1. Show that $2abc - ab - bc - ca$ is the largest integer which cannot be expressed in the form $xbc + yca + zab$,where x, y and z are non-negative integers.

1983/4. Let ABC be an equilateral triangle and $\mathcal{E}$ the set of all points contained in the three segments AB, BC and CA (including A, B and C). Determine whether, for every partition of $\mathcal{E}$ into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle. Justify your answer.

1983/5. Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression? Justify your answer.

1983/6. Let a, b and c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Determine when equality occurs.

Twenty-fifth International Olympiad, 1984

1984/1. Prove that $0 \le yz + zx + xy - 2xyz \le 7/27$, where x, y and z are non-negative real numbers for which $x + y + z = 1$.

1984/2. Find one pair of positive integers a and b such that:

(i) $ab(a + b)$ is not divisible by 7;

(ii) $(a + b)^7 - a^7 - b^7$ is divisible by 7^7 .

Justify your answer.

1984/3. In the plane two different points O and A are given. For each point X of the plane, other than O, denote by $a(X)$ the measure of the angle between OA and OX in radians, counterclockwise from $OA (0 \le a(X) < 2\pi)$. Let $C(X)$ be the circle with center O and radius of length $OX + a(X)/OX$. Each point of the plane is colored by one of a finite number of colors. Prove that there exists a point Y for which $a(Y) > 0$ such that its color appears on the circumference of the circle $C(Y)$.

1984/4. Let $ABCD$ be a convex quadrilateral such that the line CD is a tangent to the circle on AB as diameter. Prove that the line AB is a tangent to the circle on CD as diameter if and only if the lines BC and AD are parallel.

1984/5. Let d be the sum of the lengths of all the diagonals of a plane convex polygon with n vertices $(n > 3)$, and let p be its perimeter. Prove that

$$n - 3 < \frac{2d}{p} < \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] - 2,$$

where $[x]$ denotes the greatest integer not exceeding x.

1984/6. Let a, b, c and d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a + d = 2^k$ and $b + c = 2^m$ for some integers k and m, then $a = 1$.

Twenty-sixth International Olympiad, 1985

1985/1. A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

1985/2. Let n and k be given relatively prime natural numbers, $k < n$. Each number in the set $M = \{1, 2, ..., n-1\}$ is colored either blue or white. It is given that

(i) for each $i \in M$, both i and $n - i$ have the same color;

(ii) for each $i \in M, i \neq k$, both i and $|i - k|$ have the same color. Prove that all numbers in M must have the same color.

1985/3. For any polynomial $P(x) = a_0 + a_1 x + \cdots + a_k x^k$ with integer coefficients, the number of coefficients which are odd is denoted by $w(P)$. For $i = 0, 1, ...$, let $Q_i(x) = (1 + x)^i$. Prove that if $i_1 i_2, ..., i_n$ are integers such that $0 \leq i_1 < i_2 < \cdots < i_n$, then

$$w(Q_{i_1} + Q_{i_2}, + + Q_{i_n}) \geq w(Q_{i_1}).$$

1985/4. Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 26. Prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer.

1985/5. A circle with center O passes through the vertices A and C of triangle ABC and intersects the segments AB and BC again at distinct points K and N, respectively. The circumscribed circles of the triangles ABC and EBN intersect at exactly two distinct points B and M. Prove that angle OMB is a right angle.

1985/6. For every real number x_1, construct the sequence $x_1, x_2, ...$ by setting

$$x_{n+1} = x_n \left(x_n + \frac{1}{n} \right) \text{ for each } n \geq 1.$$

Prove that there exists exactly one value of x_1 for which

$$0 < x_n < x_{n+1} < 1$$

for every n.

27^{th} International Mathematical Olympiad

Warsaw, Poland

Day I

July 9, 1986

1. Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

2. A triangle $A_1A_2A_3$ and a point P_0 are given in the plane. We define $A_s = A_{s-3}$ for all $s \geq 4$. We construct a set of points P_1, P_2, P_3, $\ldots$, such that P_{k+1} is the image of P_k under a rotation with center A_{k+1} through angle 120° clockwise (for $k = 0, 1, 2, \ldots$). Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ is equilateral.

3. To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively and $y < 0$ then the following operation is allowed: the numbers x, y, z are replaced by $x + y$, $-y$, $z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to and end after a finite number of steps.

27^{th} International Mathematical Olympiad

Warsaw, Poland

Day II

July 10, 1986

4. Let A, B be adjacent vertices of a regular n-gon ($n \geq 5$) in the plane having center at O. A triangle XYZ, which is congruent to and initially conincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, X remaining inside the polygon. Find the locus of X.

5. Find all functions f, defined on the non-negative real numbers and taking non-negative real values, such that:

   (i) $f(xf(y))f(y) = f(x+y)$ for all x, $y \geq 0$,

   (ii) $f(2) = 0$,

   (iii) $f(x) \neq 0$ for $0 \leq x < 2$.

6. One is given a finite set of points in the plane, each point having integer coordinates. Is it always possible to color some of the points in the set red and the remaining points white in such a way that for any straight line L parallel to either one of the coordinate axes the difference (in absolute value) between the numbers of white point and red points on L is not greater than 1?

28^{th} International Mathematical Olympiad

Havana, Cuba

Day I

July 10, 1987

1. Let $p_n(k)$ be the number of permutations of the set $\{1, \ldots, n\}$, $n \geq 1$, which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k \cdot p_n(k) = n!.$$

(Remark: A permutation f of a set S is a one-to-one mapping of S onto itself. An element i in S is called a fixed point of the permutation f if $f(i) = i$.)

2. In an acute-angled triangle ABC the interior bisector of the angle A intersects BC at L and intersects the circumcircle of ABC again at N. From point L perpendiculars are drawn to AB and AC, the feet of these perpendiculars being K and M respectively. Prove that the quadrilateral $AKNM$ and the triangle ABC have equal areas.

3. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that for every integer $k \geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all 0, such that $|a_i| \leq k - 1$ for all i and

$$|a_1 x_1 + a_1 x_2 + \cdots + a_n x_n| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}.$$

28^{th} International Mathematical Olympiad

Havana, Cuba

Day II

July 11, 1987

4. Prove that there is no function f from the set of non-negative integers into itself such that $f(f(n)) = n + 1987$ for every n.

5. Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with rational area.

6. Let n be an integer greater than or equal to 2. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq \sqrt{n/3}$, then $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq n - 2$.

29^{th} International Mathematical Olympiad

Canberra, Australia

Day I

1. Consider two coplanar circles of radii R and r ($R > r$) with the same center. Let P be a fixed point on the smaller circle and B a variable point on the larger circle. The line BP meets the larger circle again at C. The perpendicular l to BP at P meets the smaller circle again at A. (If l is tangent to the circle at P then $A = P$.)

   (i) Find the set of values of $BC^2 + CA^2 + AB^2$.

   (ii) Find the locus of the midpoint of BC.

2. Let n be a positive integer and let $A_1, A_2, \ldots, A_{2n+1}$ be subsets of a set B. Suppose that

   (a) Each A_i has exactly $2n$ elements,

   (b) Each $A_i \cap A_j$ $(1 \leq i < j \leq 2n+1)$ contains exactly one element, and

   (c) Every element of B belongs to at least two of the A_i.

   For which values of n can one assign to every element of B one of the numbers 0 and 1 in such a way that A_i has 0 assigned to exactly n of its elements?

3. A function f is defined on the positive integers by

$$
\begin{aligned}
f(1) &= 1, \quad f(3) = 3, \\
f(2n) &= f(n), \\
f(4n+1) &= 2f(2n+1) - f(n), \\
f(4n+3) &= 3f(2n+1) - 2f(n),
\end{aligned}
$$

   for all positive integers n.

   Determine the number of positive integers n, less than or equal to 1988, for which $f(n) = n$.

29^{th} International Mathematical Olympiad

Canberra, Australia

Day II

4. Show that set of real numbers x which satisfy the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose lengths is 1988.

5. ABC is a triangle right-angled at A, and D is the foot of the altitude from A. The straight line joining the incenters of the triangles ABD, ACD intersects the sides AB, AC at the points K, L respectively. S and T denote the areas of the triangles ABC and AKL respectively. Show that $S \geq 2T$.

6. Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that

$$\frac{a^2+b^2}{ab+1}$$

is the square of an integer.

30^{th} International Mathematical Olympiad

Braunschweig, Germany

Day I

1. Prove that the set $\{1, 2, \ldots, 1989\}$ can be expressed as the disjoint union of subsets A_i ($i = 1, 2, \ldots, 117$) such that:

   (i) Each A_i contains 17 elements;

   (ii) The sum of all the elements in each A_i is the same.

2. In an acute-angled triangle ABC the internal bisector of angle A meets the circumcircle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that:

   (i) The area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$.

   (ii) The area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC.

3. Let n and k be positive integers and let S be a set of n points in the plane such that

   (i) No three points of S are collinear, and

   (ii) For any point P of S there are at least k points of S equidistant from P.

   Prove that:

$$k < \frac{1}{2} + \sqrt{2n}.$$

30^{th} International Mathematical Olympiad

Braunschweig, Germany

Day II

4. Let $ABCD$ be a convex quadrilateral such that the sides AB, AD, BC satisfy $AB = AD + BC$. There exists a point P inside the quadrilateral at a distance h from the line CD such that $AP = h + AD$ and $BP = h + BC$. Show that:
$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}.$$

5. Prove that for each positive integer n there exist n consecutive positive integers none of which is an integral power of a prime number.

6. A permutation $(x_1, x_2, \ldots, x_m)$ of the set $\{1, 2, \ldots, 2n\}$, where n is a positive integer, is said to have property P if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \ldots, 2n - 1\}$. Show that, for each n, there are more permutations with property P than without.

31^{st} International Mathematical Olympiad

Beijing, China

Day I

July 12, 1990

1. Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent line at E to the circle through D, E, and M intersects the lines BC and AC at F and G, respectively. If
$$\frac{AM}{AB} = t,$$
find
$$\frac{EG}{EF}$$
in terms of t.

2. Let $n \geq 3$ and consider a set E of $2n - 1$ distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is "good" if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.

3. Determine all integers $n > 1$ such that
$$\frac{2^n + 1}{n^2}$$
is an integer.

31^{st} International Mathematical Olympiad

Beijing, China

Day II

July 13, 1990

4. Let $\mathbb{Q}^+$ be the set of positive rational numbers. Construct a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that
$$f(xf(y)) = \frac{f(x)}{y}$$
for all x, y in $\mathbb{Q}^+$.

5. Given an initial integer $n_0 > 1$, two players, $\mathcal{A}$ and $\mathcal{B}$, choose integers n_1, n_2, n_3, ... alternately according to the following rules:

   Knowing n_{2k}, $\mathcal{A}$ chooses any integer n_{2k+1} such that
$$n_{2k} \leq n_{2k+1} \leq n_{2k}^2.$$
   Knowing n_{2k+1}, $\mathcal{B}$ chooses any integer n_{2k+2} such that
$$\frac{n_{2k+1}}{n_{2k+2}}$$
   is a prime raised to a positive integer power.

   Player $\mathcal{A}$ wins the game by choosing the number 1990; player $\mathcal{B}$ wins by choosing the number 1. For which n_0 does:

   (a) $\mathcal{A}$ have a winning strategy?

   (b) $\mathcal{B}$ have a winning strategy?

   (c) Neither player have a winning strategy?

6. Prove that there exists a convex 1990-gon with the following two properties:

   (a) All angles are equal.

   (b) The lengths of the 1990 sides are the numbers 1^2, 2^2, 3^2, ..., 1990^2 in some order.

33^{rd} International Mathematical Olympiad

First Day - Moscow - July 15, 1992
Time Limit: $4\frac{1}{2}$ hours

1. Find all integers a, b, c with $1 < a < b < c$ such that

$$(a-1)(b-1)(c-1) \quad \text{is a divisor of}\, abc-1.$$

2. Let $\mathbf{R}$ denote the set of all real numbers. Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f\left(x^2 + f(y)\right) = y + (f(x))^2 \quad \text{for all}\, x, y \in R.$$

3. Consider nine points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either colored blue or red or left uncolored. Find the smallest value of n such that whenever exactly n edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.

33^{rd} International Mathematical Olympiad

Second Day - Moscow - July 15, 1992
Time Limit: $4\frac{1}{2}$ hours

1. In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exists two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR.

2. Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane, respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|,$$

where $|A|$ denotes the number of elements in the finite set $|A|$. (Note: The orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane.)

3. For each positive integer n, $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive squares.

   (a) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.

   (b) Find an integer n such that $S(n) = n^2 - 14$.

   (c) Prove that there are infintely many integers n such that $S(n) = n^2 - 14$.

34nd International Mathematical Olympiad

First Day —July 18, 1993
Time Limit: $4\frac{1}{2}$ hours

1. Let $f(x) = x^n + 5x^{n-1} + 3$, where $n > 1$ is an integer. Prove that $f(x)$ cannot be expressed as the product of two nonconstant polynomials with integer coefficients.

2. Let D be a point inside acute triangle ABC such that $\angle ADB = \angle ACB + \pi/2$ and $AC \cdot BD = AD \cdot BC$.

   (a) Calculate the ratio $(AB \cdot CD)/(AC \cdot BD)$.

   (b) Prove that the tangents at C to the circumcircles of $\triangle ACD$ and $\triangle BCD$ are perpendicular.

3. On an infinite chessboard, a game is played as follows. At the start, n^2 pieces are arranged on the chessboard in an n by n block of adjoining squares, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is removed.

   Find those values of n for which the game can end with only one piece remaining on the board.

Second Day —July 19, 1993
Time Limit: $4\frac{1}{2}$ hours

1. For three points P, Q, R in the plane, we define $m(PQR)$ as the minimum length of the three altitudes of $\triangle PQR$. (If the points are collinear, we set $m(PQR) = 0$.)

   Prove that for points A, B, C, X in the plane,

$$m(ABC) \leq m(ABX) + m(AXC) + m(XBC).$$

2. Does there exist a function $f : \mathbf{N} \to \mathbf{N}$ such that $f(1) = 2$, $f(f(n)) = f(n) + n$ for all $n \in \mathbf{N}$, and $f(n) < f(n+1)$ for all $n \in \mathbf{N}$?

3. There are n lamps $L_0, \ldots, L_{n-1}$ in a circle ($n > 1$), where we denote $L_{n+k} = L_k$. (A lamp at all times is either on or off.) Perform steps $s_0, s_1, \ldots$ as follows: at step s_i, if L_{i-1} is lit, switch L_i from on to off or vice versa, otherwise do nothing. Initially all lamps are on. Show that:

   (a) There is a positive integer $M(n)$ such that after $M(n)$ steps all the lamps are on again;

   (b) If $n = 2^k$, we can take $M(n) = n^2 - 1$;

   (c) If $n = 2^k + 1$, we can take $M(n) = n^2 - n + 1$.

The 35th International Mathematical Olympiad (July 13-14, 1994, Hong Kong)

1. Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \le n$ for some i, j, $1 \le i \le j \le m$, there exists k, $1 \le k \le m$, with $a_i + a_j = a_k$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_m}{m} \ge \frac{n+1}{2}.$$

2. ABC is an isosceles triangle with $AB = AC$. Suppose that

1. M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB;
2. Q is an arbitrary point on the segment BC different from B and C;
3. E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear.

Prove that OQ is perpendicular to EF if and only if $QE = QF$.

3. For any positive integer k, let $f(k)$ be the number of elements in the set $\{k+1, k+2, \ldots, 2k\}$ whose base 2 representation has precisely three 1s.

- (a) Prove that, for each positive integer m, there exists at least one positive integer k such that $f(k) = m$.
- (b) Determine all positive integers m for which there exists exactly one k with $f(k) = m$.

4. Determine all ordered pairs (m, n) of positive integers such that

$$\frac{n^3 + 1}{mn - 1}$$

is an integer.

5. Let S be the set of real numbers strictly greater than -1. Find all functions $f : S \to S$ satisfying the two conditions:

1. $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x and y in S;
2. $\frac{f(x)}{x}$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$.

6. Show that there exists a set A of positive integers with the following property: For any infinite set S of primes there exist two positive integers $m \in A$ and $n \notin A$ each of which is a product of k distinct elements of S for some $k \ge 2$.

36^{th} International Mathematical Olympiad

First Day - Toronto - July 19, 1995
Time Limit: $4\frac{1}{2}$ hours

1. Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

2. Let a, b, c be positive real numbers such that $abc = 1$. Prove that
$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

3. Determine all integers $n > 3$ for which there exist n points $A_1, \ldots, A_n$ in the plane, no three collinear, and real numbers $r_1, \ldots, r_n$ such that for $1 \leq i < j < k \leq n$, the area of $\triangle A_iA_jA_k$ is $r_i + r_j + r_k$.

36^{th} International Mathematical Olympiad

Second Day - Toronto - July 20, 1995
Time Limit: $4\frac{1}{2}$ hours

1. Find the maximum value of x_0 for which there exists a sequence $x_0, x_1 \ldots, x_{1995}$ of positive reals with $x_0 = x_{1995}$, such that for $i = 1, \ldots, 1995$,
$$x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}.$$

2. Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = \pi/3$. Suppose G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = 2\pi/3$. Prove that $AG + GB + GH + DH + HE \geq CF$.

3. Let p be an odd prime number. How many p-element subsets A of $\{1, 2, \ldots\ 2p\}$ are there, the sum of whose elements is divisible by p?

37^{th} International Mathematical Olympiad

Mumbai, India

Day I 9 a.m. - 1:30 p.m.

July 10, 1996

1. We are given a positive integer r and a rectangular board $ABCD$ with dimensions $|AB| = 20, |BC| = 12$. The rectangle is divided into a grid of 20×12 unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading from the square with A as a vertex to the square with B as a vertex.

   (a) Show that the task cannot be done if r is divisible by 2 or 3.

   (b) Prove that the task is possible when $r = 73$.

   (c) Can the task be done when $r = 97$?

2. Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

   Let D, E be the incenters of triangles APB, APC, respectively. Show that AP, BD, CE meet at a point.

3. Let S denote the set of nonnegative integers. Find all functions f from S to itself such that

$$f(m + f(n)) = f(f(m)) + f(n) \qquad \forall m, n \in S.$$

37^{th} International Mathematical Olympiad

Mumbai, India

Day II 9 a.m. - 1:30 p.m.

July 11, 1996

1. The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

2. Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to FA. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF, respectively, and let P denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}.$$

3. Let p, q, n be three positive integers with $p+q < n$. Let $(x_0, x_1, \ldots, x_n)$ be an $(n+1)$-tuple of integers satisfying the following conditions:

   (a) $x_0 = x_n = 0$.

   (b) For each i with $1 \leq i \leq n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.

   Show that there exist indices $i < j$ with $(i, j) \neq (0, n)$, such that $x_i = x_j$.

38^{th} International Mathematical Olympiad

Mar del Plata, Argentina

Day I

July 24, 1997

1. In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white (as on a chessboard).

   For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along edges of the squares.

   Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let

   $$f(m, n) = |S_1 - S_2|.$$

   (a) Calculate $f(m, n)$ for all positive integers m and n which are either both even or both odd.

   (b) Prove that $f(m, n) \le \frac{1}{2}\max\{m, n\}$ for all m and n.

   (c) Show that there is no constant C such that $f(m, n) < C$ for all m and n.

2. The angle at A is the smallest angle of triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T. Show that

   $$AU = TB + TC.$$

3. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions

   $$|x_1 + x_2 + \cdots + x_n| = 1$$

   and

   $$|x_i| \le \frac{n+1}{2} \qquad i = 1, 2, \ldots, n.$$

   Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

   $$|y_1 + 2y_2 + \cdots + ny_n| \le \frac{n+1}{2}.$$

38^{th} International Mathematical Olympiad

Mar del Plata, Argentina

Day II

July 25, 1997

4. An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n - 1\}$ is called a *silver* matrix if, for each $i = 1, 2, \ldots, n$, the ith row and the ith column together contain all elements of S. Show that

   (a) there is no silver matrix for $n = 1997$;

   (b) silver matrices exist for infinitely many values of n.

5. Find all pairs (a, b) of integers $a, b \geq 1$ that satisfy the equation

$$a^{b^2} = b^a.$$

6. For each positive integer n , let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways:

$$4; 2 + 2; 2 + 1 + 1; 1 + 1 + 1 + 1.$$

   Prove that, for any integer $n \geq 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$

39^{th} International Mathematical Olympiad

Taipei, Taiwan

Day I

July 15, 1998

1. In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the perpendicular bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.

2. In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that $k/a \geq (b-1)/(2b)$.

3. For any positive integer n, let $d(n)$ denote the number of positive divisors of n (including 1 and n itself). Determine all positive integers k such that $d(n^2)/d(n) = k$ for some n.

39^{th} International Mathematical Olympiad

Taipei, Taiwai

Day II

July 16, 1998

4. Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

5. Let I be the incenter of triangle ABC. Let the incircle of ABC touch the sides BC, CA, and AB at K, L, and M, respectively. The line through B parallel to MK meets the lines LM and LK at R and S, respectively. Prove that angle RIS is acute.

6. Consider all functions f from the set N of all positive integers into itself satisfying $f(t^2f(s)) = s(f(t))^2$ for all s and t in N. Determine the least possible value of $f(1998)$.

40^{th} International Mathematical Olympiad

Bucharest

Day I

July 16, 1999

1. Determine all finite sets S of at least three points in the plane which satisfy the following condition:

   for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry for S.

2. Let n be a fixed integer, with $n \geq 2$.

   (a) Determine the least constant C such that the inequality

   $$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{1 \leq i \leq n} x_i \right)^4$$

   holds for all real numbers $x_1, \cdots, x_n \geq 0$.

   (b) For this constant C, determine when equality holds.

3. Consider an $n \times n$ square board, where n is a fixed even positive integer. The board is divided into n^2 unit squares. We say that two different squares on the board are adjacent if they have a common side.

   N unit squares on the board are marked in such a way that every square (marked or unmarked) on the board is adjacent to at least one marked square.

   Determine the smallest possible value of N.

40^{th} International Mathematical Olympiad

Bucharest

Day II

July 17, 1999

4. Determine all pairs (n, p) of positive integers such that

   p is a prime,
   n not exceeded $2p$, and
   $(p-1)^n + 1$ is divisible by n^{p-1}.

5. Two circles G_1 and G_2 are contained inside the circle G, and are tangent to G at the distinct points M and N, respectively. G_1 passes through the center of G_2. The line passing through the two points of intersection of G_1 and G_2 meets G at A and B. The lines MA and MB meet G_1 at C and D, respectively.

   Prove that CD is tangent to G_2.

6. Determine all functions $f : \mathbf{R} \longrightarrow \mathbf{R}$ such that

   $$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

   for all real numbers x, y.

41st IMO 2000

Problem 1. AB is tangent to the circles $CAMN$ and $NMBD$. M lies between C and D on the line CD, and CD is parallel to AB. The chords NA and CM meet at P; the chords NB and MD meet at Q. The rays CA and DB meet at E. Prove that $PE = QE$.

Problem 2. A, B, C are positive reals with product 1. Prove that $(A - 1 + \frac{1}{B})(B - 1 + \frac{1}{C})(C - 1 + \frac{1}{A}) \leq 1$.

Problem 3. k is a positive real. N is an integer greater than 1. N points are placed on a line, not all coincident. A *move* is carried out as follows. Pick any two points A and B which are not coincident. Suppose that A lies to the right of B. Replace B by another point B' to the right of A such that $AB' = kBA$. For what values of k can we move the points arbitrarily far to the right by repeated moves?

Problem 4. 100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?

Problem 5. Can we find N divisible by just 2000 different primes, so that N divides $2^N + 1$? [N may be divisible by a prime power.]

Problem 6. $A_1A_2A_3$ is an acute-angled triangle. The foot of the altitude from A_i is K_i and the incircle touches the side opposite A_i at L_i. The line K_1K_2 is reflected in the line L_1L_2. Similarly, the line K_2K_3 is reflected in L_2L_3 and K_3K_1 is reflected in L_3L_1. Show that the three new lines form a triangle with vertices on the incircle.

42nd International Mathematical Olympiad

Washington, DC, United States of America
July 8–9, 2001

Problems

Each problem is worth seven points.

Problem 1

Let ABC be an acute-angled triangle with circumcentre O. Let P on BC be the foot of the altitude from A.

Suppose that $\angle BCA \geq \angle ABC + 30^\circ$.

Prove that $\angle CAB + \angle COP < 90^\circ$.

Problem 2

Prove that

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1$$

for all positive real numbers a, b and c.

Problem 3

Twenty-one girls and twenty-one boys took part in a mathematical contest.

- Each contestant solved at most six problems.
- For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

Problem 4

Let n be an odd integer greater than 1, and let $k_1, k_2, \ldots, k_n$ be given integers. For each of the $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$, let

$$S(a) = \sum_{i=1}^{n} k_i a_i.$$

Prove that there are two permutations b and c, $b \neq c$, such that $n!$ is a divisor of $S(b) - S(c)$.

Problem 5

In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA.

It is known that $\angle BAC = 60^\circ$ and that $AB + BP = AQ + QB$.

What are the possible angles of triangle ABC?

Problem 6

Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that

$$a\,c + b\,d = (b + d + a - c)\,(b + d - a + c).$$

Prove that $a\,b + c\,d$ is not prime.

43rd IMO 2002

Problem 1. S is the set of all (h, k) with h, k non-negative integers such that $h + k < n$. Each element of S is colored red or blue, so that if (h, k) is red and $h' \leq h, k' \leq k$, then (h', k') is also red. A type 1 subset of S has n blue elements with different first member and a type 2 subset of S has n blue elements with different second member. Show that there are the same number of type 1 and type 2 subsets.

Problem 2. BC is a diameter of a circle center O. A is any point on the circle with $\angle AOC > 60^o$. EF is the chord which is the perpendicular bisector of AO. D is the midpoint of the minor arc AB. The line through O parallel to AD meets AC at J. Show that J is the incenter of triangle CEF.

Problem 3. Find all pairs of integers $m > 2, n > 2$ such that there are infinitely many positive integers k for which $k^n + k^2 - 1$ divides $k^m + k - 1$.

Problem 4. The positive divisors of the integer $n > 1$ are $d_1 < d_2 < \ldots < d_k$, so that $d_1 = 1, d_k = n$. Let $d = d_1 d_2 + d_2 d_3 + \cdots + d_{k-1} d_k$. Show that $d < n^2$ and find all n for which d divides n^2.

Problem 5. Find all real-valued functions on the reals such that $(f(x) + f(y))((f(u) + f(v)) = f(xu - yv) + f(xv + yu)$ for all x, y, u, v.

Problem 6. $n > 2$ circles of radius 1 are drawn in the plane so that no line meets more than two of the circles. Their centers are $O_1, O_2, \cdots, O_n$. Show that $\sum_{i<j} 1/O_i O_j \leq (n-1)\pi/4$.

44th IMO 2003

Problem 1. S is the set $\{1, 2, 3, \dots, 1000000\}$. Show that for any subset A of S with 101 elements we can find 100 distinct elements x_i of S, such that the sets $\{a + x_i | a \in A\}$ are all pairwise disjoint.

Problem 2. Find all pairs (m, n) of positive integers such that $\frac{m^2}{2mn^2-n^3+1}$ is a positive integer.

Problem 3. A convex hexagon has the property that for any pair of opposite sides the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths Show that all the hexagon's angles are equal.

Problem 4. $ABCD$ is cyclic. The feet of the perpendicular from D to the lines AB, BC, CA are P, Q, R respectively. Show that the angle bisectors of ABC and CDA meet on the line AC iff $RP = RQ$.

Problem 5. Given $n > 2$ and reals $x_1 \leq x_2 \leq \cdots \leq x_n$, show that $(\sum_{i,j} |x_i - x_j|)^2 \leq \frac{2}{3}(n^2 - 1) \sum_{i,j}(x_i - x_j)^2$. Show that we have equality iff the sequence is an arithmetic progression.

Problem 6. Show that for each prime p, there exists a prime q such that $n^p - p$ is not divisible by q for any positive integer n.

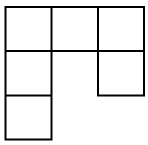
45rd IMO 2004

Problem 1. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

Problem 2. Find all polynomials f with real coefficients such that for all reals a,b,c such that $ab + bc + ca = 0$ we have the following relations

$$f(a-b) + f(b-c) + f(c-a) = 2f(a+b+c).$$

Problem 3. Define a "hook" to be a figure made up of six unit squares as shown below in the picture, or any of the figures obtained by applying rotations and reflections to this figure.



Determine all $m \times n$ rectangles that can be covered without gaps and without overlaps with hooks such that

- the rectangle is covered without gaps and without overlaps
- no part of a hook covers area outside the rectagle.

Problem 4. Let $n \geq 3$ be an integer. Let $t_1, t_2, ..., t_n$ be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + ... + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + ... + \frac{1}{t_n}\right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

Problem 5. In a convex quadrilateral $ABCD$ the diagonal BD does not bisect the angles ABC and CDA. The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \text{ and } \angle PDC = \angle BDA.$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

Problem 6. We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity.
Find all positive integers n such that n has a multiple which is alternating.

46rd IMO 2005

Problem 1. Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths.
Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

Problem 2. Let $a_1, a_2, \dots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, \dots, a_n$ leave n different remainders upon division by n.
Prove that every integer occurs exactly once in the sequence $a_1, a_2, \dots$.

Problem 3. Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

Problem 4. Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1, \ n \geq 1.$$

Problem 5. Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R.
Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

Problem 6. In a mathematical competition, in which 6 problems were posed to the participants, every two of these problems were solved by more than $\frac{2}{5}$ of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each.

12 July 2006

Problem 1. Let ABC be a triangle with incentre I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

Problem 2. Let P be a regular 2006-gon. A diagonal of P is called *good* if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called *good*.

Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

Problem 3. Determine the least real number M such that the inequality

$$\left| ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2) \right| \leq M(a^2 + b^2 + c^2)^2$$

holds for all real numbers a, b and c.

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points

13 July 2006

Problem 4. Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

Problem 5. Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x)) \ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

Problem 6. Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points

July 25, 2007

Problem 1. Real numbers $a_1, a_2, \ldots, a_n$ are given. For each i $(1 \leq i \leq n)$ define

$$d_i = \max\{a_j : 1 \leq j \leq i\} - \min\{a_j : i \leq j \leq n\}$$

and let

$$d = \max\{d_i : 1 \leq i \leq n\}.$$

(a) Prove that, for any real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$,

$$\max\{|x_i - a_i| : 1 \leq i \leq n\} \geq \frac{d}{2}. \qquad (*)$$

(b) Show that there are real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$ such that equality holds in $(*)$.

Problem 2. Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

Problem 3. In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its *size*.

Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points

Language: English

July 26, 2007

Problem 4. In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

Problem 5. Let a and b be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.

Problem 6. Let n be a positive integer. Consider

$$S = \{(x, y, z) \ : \ x, y, z \in \{0, 1, \ldots, n\},\ x + y + z > 0\}$$

as a set of $(n+1)^3 - 1$ points in three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0, 0, 0)$.

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points

Language: **English** Day: **1**

49th INTERNATIONAL MATHEMATICAL OLYMPIAD
MADRID (SPAIN), JULY 10-22, 2008

Wednesday, July 16, 2008

Problem 1. An acute-angled triangle ABC has orthocentre H. The circle passing through H with centre the midpoint of BC intersects the line BC at A_1 and A_2. Similarly, the circle passing through H with centre the midpoint of CA intersects the line CA at B_1 and B_2, and the circle passing through H with centre the midpoint of AB intersects the line AB at C_1 and C_2. Show that A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle.

Problem 2. (a) Prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

for all real numbers x, y, z, each different from 1, and satisfying $xyz = 1$.

(b) Prove that equality holds above for infinitely many triples of rational numbers x, y, z, each different from 1, and satisfying $xyz = 1$.

Problem 3. Prove that there exist infinitely many positive integers n such that $n^2 + 1$ has a prime divisor which is greater than $2n + \sqrt{2n}$.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

49th INTERNATIONAL MATHEMATICAL OLYMPIAD
MADRID (SPAIN), JULY 10-22, 2008

Thursday, July 17, 2008

Problem 4. Find all functions $f : (0, \infty) \to (0, \infty)$ (so, f is a function from the positive real numbers to the positive real numbers) such that

$$\frac{\big(f(w)\big)^2 + \big(f(x)\big)^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, z, satisfying $wx = yz$.

Problem 5. Let n and k be positive integers with $k \geq n$ and $k - n$ an even number. Let $2n$ lamps labelled $1, 2, \ldots, 2n$ be given, each of which can be either *on* or *off*. Initially all the lamps are off. We consider sequences of *steps*: at each step one of the lamps is switched (from on to off or from off to on).

Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off.

Let M be the number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off, but where none of the lamps $n + 1$ through $2n$ is ever switched on.

Determine the ratio N/M.

Problem 6. Let $ABCD$ be a convex quadrilateral with $|BA| \neq |BC|$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents of ω_1 and ω_2 intersect on ω.

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Wednesday, July 15, 2009

Problem 1. Let n be a positive integer and let $a_1, \ldots, a_k$ ($k \geq 2$) be distinct integers in the set $\{1, \ldots, n\}$ such that n divides $a_i(a_{i+1}-1)$ for $i = 1, \ldots, k-1$. Prove that n does not divide $a_k(a_1-1)$.

Problem 2. Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ, respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

Problem 3. Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1}, s_{s_2}, s_{s_3}, \ldots \qquad \text{and} \qquad s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

50th International Mathematical Olympiad
Bremen | Germany | 2009

Language: **English**

Day: **2**

Thursday, July 16, 2009

Problem 4. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incentre of triangle ADC. Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle CAB$.

Problem 5. Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a, \ f(b) \ \text{ and } \ f(b + f(a) - 1).$$

(A triangle is *non-degenerate* if its vertices are not collinear.)

Problem 6. Let $a_1, a_2, \ldots, a_n$ be distinct positive integers and let M be a set of $n - 1$ positive integers not containing $s = a_1 + a_2 + \cdots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

IMO Solutions
1970-2003 and 2006

Solution (1970-2003 and 2006)

IMO 1970

A1

We need an expression for r/q. There are two expressions, one in terms of angles and the other in terms of sides. The latter is a poor choice, because it is both harder to derive and less useful. So we derive the angle expression.

Let I be the center of the in-circle for ABC and X the center of the external circle for ABC. I is the intersection of the two angle bisectors from A and B, so c = r (cot A/2 + cot B/2). The X lies on the bisector of the external angle, so angle XAB is 90^o – A/2. Similarly, angle XBA is 90^o – B/2, so c = q (tan A/2 + tan B/2). Hence r/q = (tan A/2 + tan B/2)/(cot A/2 + cot B/2) = tan A/2 tan B/2.

Applying this to the other two triangles, we get r_1/q_1 = tan A/2 tan CMA/2, r_2/q_2 = tan B/2 tan CMB/2. But CMB/2 = 90^o – CMA/2, so tan CMB/2 = 1/tan CMA/2. Hence result.

B2

The first step is to show that angles ADB and ADC are also 90^o. Let H be the intersection of the altitudes of ABC and let CH meet AB at X. Planes CED and ABC are perpendicular and AB is perpendicular to the line of intersection CE. Hence AB is perpendicular to the plane CDE and hence to ED. So $BD^2 = DE^2 + BE^2$. Also $CB^2 = CE^2 + BE^2$. Subtracting: $CB^2 - BD^2 = CE^2 - DE^2$. But $CB^2 - BD^2 = CD^2$, so $CE^2 = CD^2 + DE^2$, so angle CDE = 90^o. But angle CDB = 90^o, so CD is perpendicular to the plane DAB, and hence angle CDA = 90^o. Similarly, angle ADB = 90^o.

Hence $AB^2 + BC^2 + CA^2 = 2(DA^2 + DB^2 + DC^2)$. But now we are done, because Cauchy's inequality gives $(AB + BC + CA)^2 \le 3(AB^2 + BC^2 + CA^2)$.

We have equality iff we have equality in Cauchy's inequality, which means AB = BC = CA.

B3

Improved and corrected by Gerhard Wöginger, Technical University Graz

At most 3 of the triangles formed by 4 points can be acute. It follows that at most 7 out of the 10 triangles formed by any 5 points can be acute. For given 10 points, the maximum no. of acute triangles is: the no. of subsets of 4 points x 3/the no. of subsets of 4 points containing 3 given points. The total no. of triangles is the same expression with the first 3 replaced by 4. Hence at most 3/4 of the 10, or 7.5, can be acute, and hence at most 7 can be acute.

The same argument now extends the result to 100 points. The maximum number of acute triangles formed by 100 points is: the no. of subsets of 5 points x 7/the no. of subsets of 5 points containing 3 given points. The total no. of triangles is the same expression with 7 replaced by 10. Hence at most 7/10 of the triangles are acute.

IMO 1973

A1

We proceed by induction on n. It is clearly true for n = 1. Assume it is true for 2n−1. Given OP_i for 2n+1, reorder them so that all OP_i lie between OP_{2n} and OP_{2n+1}. Then $u = OP_{2n} + OP_{2n+1}$ lies along the angle bisector of angle $P_{2n}OP_{2n+1}$ and hence makes an angle less than 90^o with $v = OP_1 + OP_2 + \dots + OP_{2n-1}$ (which must lie between OP_1 and OP_{2n-1} and hence between OP_{2n} and OP_{2n+1}. By induction $|v| \ge 1$. But $|u + v| \ge |v|$ (use the cosine formula). Hence the result is true for 2n+1.

It is clearly best possible: take $OP_1 = \dots = OP_n = -OP_{n+1} = \dots = -OP_{2n}$, and OP_{2n+1} in an arbitrary direction.

A2

To warm up, we may notice that a regular hexagon is a planar set satisfying the condition.

Take two regular hexagons with a common long diagonal and their planes perpendicular. Now if we take A, B in the same hexagon, then we can find C, D in the same hexagon. If we take A in one and B in the other, then we may take C at the opposite end of a long diagonal from A, and D at the opposite end of a long diagonal from B.

A3

Put $y = x + 1/x$ and the equation becomes $y^2 + ay + b - 2 = 0$, which has solutions $y = -a/2 \pm\sqrt{(a^2 + 8 - 2b)}/2$. We require $|y| \ge 2$ for the original equation to have a real root and hence we need $|a| + \sqrt{(a^2 + 8 - 4b)} \ge 4$. Squaring gives $2|a| - b \ge 2$. Hence $a^2 + b^2 \ge a^2 + (2 - 2|a|)^2 = 5a^2 - 8|a| + 4 = 5(|a| - 4/5)^2 + 4/5$. So the least possible value of $a^2 + b^2$ is 4/5, achieved when a = 4/5, b = −2/5. In this case, the original equation is $x^4 + 4/5\, x^3 - 2/5\, x^2 + 4/5\, x + 1 = (x + 1)^2(x^2 - 6/5\, x + 1)$.

B1

In particular he must sweep the other two vertices. Let us take the triangle to be ABC, with side 1 and assume the soldier starts at A. So the path must intersect the circles radius $\sqrt{3}/4$ centered on the other two vertices. Let us look for the shortest path of this type. Suppose it intersects the circle center B at X and the circle center C at Y, and goes first to X and then to Y. Clearly the path from A to X must be a straight line and the path from X to Y must be a straight line. Moreover the shortest path from X to the circle center C follows the line XC and has length AX + XC − $\sqrt{3}/4$. So we are looking for the point X which minimises AX + XC.

Consider the point P where the altitude intersects the circle. By the usual reflection argument the distance AP + PC is shorter than the distance AP' + P'C for any other point P' on the line perpendicular to the altitude through P. Moreover for any point X on the circle, take AX to cut the line at P'. Then AX + XC > AP' + P'C > AP + PC.

It remains to check that the three circles center A, X, Y cover the triangle. In fact the circle center X covers the whole triangle except for a small portion near A and a small portion near C, which are covered by the triangles center A and Y.

B2

f(x) = ax + b has fixed point b/(1−a). If a = 1, then b must be 0, and any point is a fixed point. So suppose f(x) = ax + b and g(x) = ax + b' are in G. Then h the inverse of f is given by h(x) = x/a − b/a, and hg(x) = x + b'/a − b/a. This is in G, so we must have b' = b.

Suppose f(x) = ax + b, and g(x) = cx + d are in G. Then fg(x) = acx + (ad + b), and gf(x) = acx + (bc + d). We must have ad + b = bc + d and hence b/(1−a) = c/(1−d), in other words f and g have the same fixed point.

B3

We notice that the constraints are linear, in the sense that if b_i is a solution for a_i, q, and b_i' is a solution for a_i', q, then for any k, k' > 0 a solution for $ka_i + k'a_i'$, q is $kb_i + k'b_i'$. Also a "near" solution for $a_h = 1$, other $a_i = 0$ is $b_1 = q^{h-1}$, $b_2 = q^{h-2}$, ... , $b_{h-1} = q$, $b_h = 1$, $b_{h+1} = q$, ... , $b_n = q^{n-h}$. "Near" because the inequalities in (a) and (b) are not strict.

However, we might reasonably hope that the inequalities would become strict in the linear combination, and indeed that is true. Define $b_r = q^{r-1}a_1 + q^{r-2}a_2 + \dots + qa_{r-1} + a_r + qa_{r+1} + \dots + q^{n-r}a_n$. Then we may easily verify that (a) − (c) hold.

IMO 1974

A1

The player with 9 counters.

The total of the scores, 39, must equal the number of rounds times the total of the cards. But 39 has no factors except 1, 3, 13 and 39, the total of the cards must be at least 1 + 2 + 3 = 6, and the number of rounds is at least 2. Hence there were 3 rounds and the cards total 13.

The highest score was 20, so the highest card is at least 7. The score of 10 included at least one highest card, so the highest card is at most 8. The lowest card is at most 2, because if it was higher then the highest card would be at most 13 – 3 – 4 = 6, whereas we know it is at least 7. Thus the possibilities for the cards are: 2, 3, 8; 2, 4, 7; 1, 4, 8; 1, 5, 7. But the only one of these that allows a score of 20 is 1, 4, 8. Thus the scores were made up: 8 + 8 + 4 = 20, 8 + 1 + 1 = 10, 4 + 4 + 1 = 9. The last round must have been 4 to the player with 20, 8 to the player with 10 and 1 to the player with 9. Hence on each of the other two rounds the cards must have been 8 to the player with 20, 1 to the player with 10 and 4 to the player with 9.

A2

Extend CD to meet the circumcircle of ABC at E. Then CD·DE = AD·DB, so CD is the geometric mean of AD and DB iff CD = DE. So we can find such a point iff the distance of C from AB is less than the distance of AB from the furthest point of the arc AB on the opposite side of AB to C. The furthest point F is evidently the midpoint of the arc AB. F lies on the angle bisector of C. So ∠FAB = ∠FAC = ∠C/2. Hence distance of F from AB is c/2 tan C/2 (as usual we set c = AB, b = CA, a = BC). The distance of C from AB is a sin B. So a necessary and sufficient condition is c/2 tan C/2 ≥ a sin B. But by the sine rule, a = c sin A/sin C, so the condition becomes (sin C/2 sin C)/(2 cos C/2) ≥ sin A sin B. But sin C = 2 sin C/2 cos C/2, so we obtain the condition quoted in the question.

A3

Let k = √8. Then $(1 + k)^{2n+1}$ = a + bk, where b is the sum given in the question. Similarly, $(1 - k)^{2n+1}$ = a – bk. This looks like a dead end, because eliminating a gives an unhelpful expression for b. The trick is to multiply the two expressions to get $7^{2n+1} = 8b^2 - a^2$. This still looks unhelpful, but happens to work, because we soon find that $7^{2n+1} \neq \pm 2$ (mod 5). So if b was a multiple of 5 then we would have a square congruent to ±2 (mod 5) which is impossible.

B1

The requirement that the number of black and white squares be equal is equivalent to requiring that the each rectangle has an even number of squares. $2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 = 72 > 64$, so $p < 8$. There are 5 possible divisions of 64 into 7 unequal even numbers: 2 + 4 + 6 + 8 + 10 + 12 + 22; 2 + 4 + 6 + 8 + 10 + 16 + 18; 2 + 4 + 6 + 8 + 12 + 14 + 18; 2 + 4 + 6 + 10 + 12 + 14 + 16. The first is ruled out because a rectangle with 22 squares would have more than 8 squares on its longest side. The others are all possible.

2 2 2 2 2 2 2 4   2 2 2 2 2 2 2 2

2 2 2 2 2 2 2 4   2 2 2 2 2 2 2 2

1 1 1 1 1 5 5 4   1 1 1 1 1 1 5 5

1 1 1 1 1 5 5 4   1 1 1 1 1 1 5 5

1 1 1 1 1 5 5 4   1 1 1 1 1 1 5 5

1 1 1 1 1 6 6 4   3 3 3 3 3 7 6 6

3 3 3 3 3 6 6 4   3 3 3 3 3 7 6 6

3 3 3 3 3 7 7 4   4 4 4 4 4 4 4 4

2 2 2 2 2 2 2 7   1 1 1 1 1 1 1 1

2 2 2 2 2 2 2 7   1 1 1 1 1 1 1 1

1 1 1 1 1 1 4 4   2 2 2 2 2 2 2 7

1 1 1 1 1 1 4 4   2 2 2 2 2 2 2 7

1 1 1 1 1 1 4 4   3 3 3 3 3 3 6 6

3 3 3 3 3 3 4 4   3 3 3 3 3 3 6 6

3 3 3 3 3 3 6 6   4 4 4 4 4 5 5 5

5 5 5 5 5 5 6 6   4 4 4 4 4 5 5 5

B2

We show first that the sum must lie between 1 and 2. If we replace each denominator by a+b+c+d then we reduce each term and get 1. Hence the sum is more than 1. Suppose a is the largest of the four reals. Then the first term is less than 1. The second and fourth terms have denominators greater than b+c+d, so the terms are increased if we replace the denominators by b+c+d. But then the last three terms sum to 1. Thus the sum of the last three terms is less than 1. Hence the sum is less than 2.

If we set a = c = 1 and make b and d small, then the first and third terms can be made arbitarily close to 1 and the other two terms arbitarily close to 0, so we can make the sum arbitarily close to 2. If we set a = 1, c = d and make b and c/b arbitarily small, then the first term is arbitarily close to 1 and the last three terms are all arbitarily small, so we can make the sum arbitarily close to 1. Hence, by continuity, we can achieve any value in the open interval (1,2).

B3

Suppose that A(x) and B(x) are two polynomials with integer coefficients which are identical except for their constant terms, which differ by 2. Suppose A(r) = 0, and B(s) =0 with r and s integers. Then subtracting we get 2 plus a sum of terms $a(r^i - s^i)$. Each of these terms is divisible by (r – s), so 2 must be divisible by (r – s). Hence r and s differ by 0, 1 or 2.

Now let r be the smallest root of P(x) = 1 and P(x) = −1. The polynomial with r as a root can have at most d distinct roots and hence at most d distinct integer roots. If s is a root of the other equation then s must differ from r by 0, 1, or 2. But $s \geq r$, so s = r, r+1 or r+2. Hence the other equation adds at most 2 distinct integer roots.

IMO 1975

A1

If $x \geq x'$ and $y \geq y'$, then $(x - y)^2 + (x' - y')^2 \leq (x - y')^2 + (x' - y)^2$. Hence if i < j, but $z_i \leq z_j$, then swapping z_i and z_j reduces the sum of the squares. But we can return the order of the z_i to y_i by a sequence of swaps of this type: first swap 1 to the 1st place, then 2 to the 2nd place and so on.

A2

We must be able to find a set S of infinitely many a_n in some residue class mod a_i. Take a_j to be a member of S. Then for any a_n in S satisfying $a_n > a_j$, we have $a_n = a_j$ + a multiple of a_i.

A3

Trigonometry provides a routine solution. Let BC = a, CA = b, AB = c. Then, by the sine rule applied to AQC, AQ = b/(2 sin 105°) = b/(2 cos 15°). Similarly, PB = a/(2 cos 15). Also AR = RB = c/(2 cos 15°). So by the cosine rule $RP^2 = (a^2 + c^2 - 2ac \cos(B+60°))/(4 \cos^2 15°)$, and $RQ^2 = (b^2 + c^2 - 2bc \cos(A+60°))/(4 \cos^2 15°)$. So RP = RQ is equivalent to: $a^2 - 2ac \cos(60°+B) = b^2 - 2bc \cos(60°+A)$ and hence to $a^2 - ac \cos B + \sqrt{3}\, ac \sin B = b^2 - bc \cos A + \sqrt{3}\, bc \sin A$. By the sine rule, the sine terms cancel. Also b – b cos A = a cos C, and a – c cos B = b cos C, so the last equality is true and hence RP = RQ. We get an exactly similar expression for PQ^2 and show that it equals 2 RP^2 in the same way.

A more elegant solution is to construct S on the outside of AB so that ABS is equilateral. Then we find that CAS and QAR are similar and that CBS and PBR are similar. So QR/CS = PR/CS. The ratio of the sides is the same in each case (CA/QA = CB/PB since CQA and CPB are similar), so QR = PR. Also there is a 45° rotation between QAR and CAS and another 45° rotation between CBS and PBR, hence QR and PR are at 90°.

B1

Let $X = 4444^{4444}$.Then X has less than 4.4444 = 17776 digits, so A is at most 9.17776 = 159984. Hence B is at most 6.9 = 54. But all these numbers are congruent mod 9. 4444 = –2 (mod 9), so $X = (-2)^{4444}$ (mod 9). But $(-2)^3 = 1$ (mod 9), and 4444 = 1 (mod 3), so X = –2 = 7 (mod 9). But any number less than 55 and congruent to 7 has digit sum 7 (possibilities are 7, 16, 25, 34, 43, 52). Hence the answer is 7.

B2

Let x be the angle $\cos^{-1} 4/5$, so that cos x = 4/5, sin x = 3/5. Take points on the unit circle at angles 2nx for n integral. Then the distance between the points at angles 2nx and 2mx is 2 sin(n – m)x. The usual formula, giving sin(n – m)x in terms of sin x and cos x, shows that sin(n – m)x is rational. So it only remains to show that this process generates arbitarily many distinct points, in other words that x is not a rational multiple of π.

This is quite hard. There is an elegant argument in sections 5 and 8 of Hadwiger et al, Combinatorial geometry in the Plane. But we can avoid it by observing that there are only finitely many numbers with are nth roots of unity for $n \le 2 \times 1975$, whereas there are infinitely many Pythagorean triples, so we simply pick a triple which is not such a root of unity.

B3

(1) means that P is homogeneous of degree n for some n. Experimenting with low n, shows that the only solutions for n = 1, 2, 3 are $(x - 2y)$, $(x + y)(x - 2y)$, $(x + y)^2(x - 2y)$. It then obvious by inspection that $(x + y)^n(x - 2y)$ is a solution for any n. Taking x = y = z in (2) shows that P(2x,x) = 0, so (x − 2y) is always a factor. Taking x = y = 1, z = −2 gives $P(1,-1)\ (2^n - 2) = 0$, so (x + y) is a factor for n > 1. All this suggests (but does not prove) that the general solution is $(x + y)^n(x - 2y)$.

Take y = 1 − x, z = 0 in (2) and we get: P(x, 1−x) = −1 − P(1−x, x). In particular, P(0,1) = −2. Now take z = 1 − x − y and we get: P(1−x, x) + P(1−y, y) + P(x+y, 1−x−y) = 0 and hence f(x+y) = f(x) + f(y), where f(x) = P(1−x, x) − 1. By induction we conclude that, for any integer m and real x, f(mx) = mf(x). Hence f(1/s) = 1/s f(1) and f(r/s) = r/s f(1) for any integers r, s. But P(0,1) = −2, so f(1) = −3. So f(x) = −3x for all rational x. But f is continuous, so f(x) = −3x for all x. So set x = b/(a+b), where a and b are arbitrary reals (with a+b non-zero). Then $P(a,b) = (a+b)^nP(1-x, x) = (a+b)^n(-3b/(a+b) + 1) = (a+b)^{n-1}(a-2b)$, as claimed. [For a+b = 0, we appeal to continuity, or use the already derived fact that for n > 1, P(a,b) = 0.]

IMO 1976

A1

At first sight, the length of the other diagonal appears unlikely to be significantly constrained. However, a little experimentation shows that it is hard to get such a low value as 16. This suggests that 16 may be the smallest possible value.

If the diagonal which is part of the 16 has length x, then the area is the sum of the areas of two triangles base x, which is xy/2, where y is the sum of the altitudes of the two triangles. y must be at most (16 − x), with equality only if the two triangles are right-angled. But $x(16 - x)/2 = (64 - (x - 8)^2)/2 \le 32$ with equality only iff x = 8. Thus the only way we can achieve the values given is with one diagonal length 8 and two sides perpendicular to this diagonal with lengths totalling 8. But in this case the other diagonal has length $8\sqrt{2}$.

A2

We show that the graph of P_n can be divided into 2^n lines each joining the top and bottom edges of the square side 4 centered on the origin (vertices (2,2), (−2,2), (−2,−2), (−2,2)). We are then home because the upward sloping diagonal of the square, which represents the graph of y = x, must cut each of these lines and hence give 2^n distinct real roots of $P_n(x) = x$ in the range [−2,2]. But P_n is a polynomial of degree 2^n, so it has exactly 2^n roots. Hence all its roots are real and distinct.

We prove the result about the graph by induction. It is true for n = 1: the first line is the graph from x = –2 to 0, and the second line is the graph from 0 to 2. So suppose it is true for n. Then P_1 turns each of the 2^n lines for P_n into two lines for P_{n+1}, so the result is true for n+1.

Alternative solution from Arthur Engel, Problem-Solving Strategies, Springer 1998 [Problem books in mathematics series], ISBN 0387982191. A rather good training book.

Put x = 2 cos t (so we are restricting attention to $-2 \le x \le 2$). Then we find $P_n(x) = 2 \cos 2^n t$, so the equation $P_n(x) = x$ becomes $\cos 2^n t = \cos t$. By inspection, has the 2^n solutions $t = 2k\pi/(2^n - 1)$ and $t = 2k\pi/(2^n + 1)$, giving 2^n distinct solutions in x.

A3

Answer: 2 x 3 x 5 or 2 x 5 x 6.

This is somewhat messy. The basic idea is that the sides cannot be too long, because then the ratio becomes too big. Let k denote the (real) cube root of 2. Given any integer n, let n' denote the least integer such that n'k <= n. Let the sides of the box be $a \le b \le c$. So we require 5a'b'c' = abc (*).

It is useful to derive n' for small n: 1' = 0, 2' = 1, 3' = 2, 4' = 3, 5' = 3, 6' = 4, 7' = 5, 8' = 6, 9' = 7, 10' = 7.

Clearly $n'k \ge n-2$. But $6^3 > 0.4\ 8^3$, and hence $(n'k)^3 \ge (n - 2)^3 > 0.4\ n^3$ for all $n \ge 8$. We can check directly that $(n'k)^3 > 0.4\ n^3$ for n = 3, 4, 5, 6, 7. So we must have a = 2 (we cannot have a = 1, because 1' = 0).

From (*) we require b or c to be divisible by 5. Suppose we take it to be 5. Then since 5' = 3, the third side n must satisfy: n' = 2/3 n. We can easily check that 2k/3 < 6/7 and hence (2/3 nk + 1) < n for $n \ge 7$, so n' > 2/3 n for $n \ge 7$. This just leaves the values n = 3 and n = 6 to check (since n' = 2/3 n is integral so n must be a multiple of 3). Referring to the values above, both these work. So this gives us two possible boxes: 2 x 3 x 5 and 2 x 5 x 6.

The only remaining possibility is that the multiple of 5 is at least 10. But then it is easy to check that if it is m then $m'/m \ge 7/10$. It follows from (*) that the third side r must satisfy r'/r <= 4/7. But using the limit above and referring to the small values above, this implies that r must be 2. So a = b = 2. But now c must satisfy c' = 4/5 c. However, that is impossible because 4/5 k > 1.

B1

Answer: $2 \cdot 3^{658}$.

There cannot be any integers larger than 4 in the maximal product, because for $n > 4$, we can replace n by 3 and $n - 3$ to get a larger product. There cannot be any 1s, because there must be an integer $r > 1$ (otherwise the product would be 1) and $r + 1 > 1.r$. We can also replace any 4s by two 2s leaving the product unchanged. Finally, there cannot be more than two 2s, because we can replace three 2s by two 3s to get a larger product. Thus the product must consist of 3s, and either zero, one or two 2s. The number of 2s is determined by the remainder on dividing the number 1976 by 3.

$1976 = 3 \cdot 658 + 2$, so there must be just one 2, giving the product $2 \cdot 3^{658}$.

B2

We use a counting argument. If the modulus of each x_i is at most n, then each of the linear combinations has a value between $-2n^2$ and $2n^2$, so there are at most $(4n^2 + 1)$ possible values for each linear combination and at most $(2n^2 + 1)^n$ possible sets of values. But there are 2n+1 values for each x_i with modulus at most n, and hence $(2n+1)^{2n} = (4n^2+4n+1)^n$ sets of values. So two distinct sets must give the same set of values for the linear combinations. But now if these sets are x_i and x_i', then the values $x_i - x_i'$ give zero for each linear combination, and have modulus at most 2n. Moreover they are not all zero, since the two sets of values were distinct.

B3

Experience with recurrence relations suggests that the solution is probably the value given for $[u_n]$ plus its inverse. It is straightforward to verify this guess by induction.

Squaring u_{n-1} gives the sum of positive power of 2, its inverse and 2. So $u_{n-1} - 2$ = the sum of a positive power of 2 and its inverse. Multiplying this by u_n gives a positive power of 2 + its inverse + 2 + 1/2, and we can check that the power of 2 is correct for u_{n+1}.

IMO 1977

A1

The most straightforward approach is to use coordinates. Take A, B, C, D to be (1,1), (−1,1), (−1,−1), (1,−1). Then K, L, M, N are (0, −2k), (2k, 0), (0, 2k), (−2k, 0), where $k = (\sqrt{3} - 1)/2$. The midpoints of KL, LM, MN, NK are (k, −k), (k, k), (−k, k), (−k, −k). These are all a distance $k\sqrt{2}$ from the origin, at angles 315, 45, 135, 225 respectively. The midpoints of AK, BK, BL, CL, CM, DM, DN, AN are (h, j), (−h, j), (−j, h), (−j, −h), (−h, −j), (h, −j), (j, −h), (j, h), where h = 1/2, j =

($1 - 1/2\ \sqrt{3}$). These are also at a distance $k\sqrt{2}$ from the origin, at angles 15, 165, 105, 255, 195, 345, 285, 75 respectively. For this we need to consider the right-angled triangle sides k, h, j. The angle x between h and k has sin x = j/k and cos x = h/k. So sin 2x = 2 sin x cos x = $2hj/k^2$ = 1/2. Hence x = 15.

So the 12 points are all at the same distance from the origin and at angles 15 + 30n, for n = 0, 1, 2, ... , 11. Hence they form a regular dodecagon.

A2

Answer: 16.

$x_1 + \dots + x_7 < 0$, $x_8 + \dots + x_{14} < 0$, so $x_1 + \dots + x_{14} < 0$. But $x_4 + \dots + x_{14} > 0$, so $x_1 + x_2 + x_3 < 0$. Also $x_5 + \dots + x_{11} < 0$ and $x_1 + \dots + x_{11} > 0$, so $x_4 > 0$. If there are 17 or more elements then the same argument shows that $x_5, x_6, x_7 > 0$. But $x_1 + \dots + x_7 < 0$, and $x_5 + \dots + x_{11} < 0$, whereas $x_1 + \dots + x_{11} > 0$, so $x_5 + x_6 + x_7 < 0$. Contradiction.

If we assume that there is a solution for n = 16 and that the sum of 7 consecutive terms is −1 and that the sum of 11 consecutive terms is 1, then we can easily solve the equations to get: 5, 5, −13, 5, 5, 5, −13, 5, 5, −13, 5, 5, 5, −13, 5, 5 and we can check that this works for 16.

A3

Take a, b, c, d = −1 (mod n). The idea is to take abcd which factorizes as ab.cd or ac.bd. The hope is that ab, cd, ac, bd will not factorize in V_n. But a little care is needed, since this is not necessarily true.

Try taking a = b = n − 1, c = d = 2n −1. a^2 must be indecomposable because it is less than the square of the smallest element in V_n. If $ac = 2n^2 - 3n + 1$ is decomposable, then we have kk'n + k + k' = 2n − 3 for some k, k' >= 1. But neither of k or k' can be 2 or more, because then the lhs is too big, and k = k' = 1 does not work unless n = 5. Similarly, if c^2 is decomposable, then we have kk'n + k + k' = 4n − 4. k = k' = 1 only works for n = 2, but we are told n > 2. k = 1, k' = 2 does not work (it would require n = 7/2). k = 1, k' = 3 only works for n = 8. Other possibilities make the lhs too big.

So if n is not 5 or 8, then we can take the number to be $(n - 1)^2(2n - 1)^2$, which factors as $(n - 1)^2 \times (2n - 1)^2$ or as (n − 1)(2n − 1) x (n − 1)(2n − 1). This does not work for 5 or 8: 16·81 = 36·36, but 36 decomposes as 6·6; 49·225 = 105·105, but 225 decomposes as 9·25.

For n = 5, we can use 3136 = 16·196 = 56·56. For n = 8, we can use 25921 = 49·529 = 161·161.

B1

Take y so that cos y = $a/\sqrt{(a^2 + b^2)}$, sin y = $b/\sqrt{(a^2 + b^2)}$, and z so that cos 2z = $A/\sqrt{(A^2 + B^2)}$, sin 2z = $B/\sqrt{(A^2 + B^2)}$. Then f(x) = 1 − c cos(x − y) − C cos2(x − z), where $c = \sqrt{(a^2 + b^2)}$, $C = \sqrt{(A^2 + B^2)}$.

f(z) + f(π + z) ≥ 0 gives C ≤ 1. f(y + π/4) + f(y − π/4) ≥ 0 gives c ≤ √2.

B2

$a^2 + b^2 >= (a + b)^2/2$, so q ≥ (a + b)/2. Hence r < 2q. The largest square less than 1977 is $1936 = 44^2$. $1977 = 44^2 + 41$. The next largest gives $1977 = 43^2 + 128$. But 128 > 2.43. So we must have q = 44, r = 41. Hence $a^2 + b^2 = 44(a + b) + 41$, so $(a - 22)^2 + (b - 22)^2 = 1009$. By trial, we find that the only squares with sum 1009 are 28^2 and 15^2. This gives two solutions 50, 37 or 50, 7.

B3

The first step is to show that f(1) < f(2) < f(3) < ... . We do this by induction on n. We take S_n to be the statement that f(n) is the unique smallest element of { f(n), f(n+1), f(n+2), ... }.

For m > 1, f(m) > f(s) where s = f(m−1), so f(m) is not the smallest member of the set {f(1), f(2), f(3), ... }. But the set is bounded below by zero, so it must have a smallest member. Hence the unique smallest member is f(1). So S_1 is true.

Suppose S_n is true. Take m > n+1. Then m−1 > n, so by S_n, f(m−1) > f(n). But S_n also tells us that f(n) > f(n−1) > ... > f(1), so f(n) ≥ n − 1 + f(1) ≥ n. Hence f(m−1) ≥ n+1. So f(m−1) belongs to { n+1, n+2, n+3, .. }. But we are given that f(m) > f(f(m−1)), so f(m) is not the smallest element of { f(n+1), f(n+2), f(n+3), ... }. But there must be a smallest element, so f(n+1) must be the unique smallest member, which establishes S_{n+1}. So, S_n is true for all n.

So n ≤ m implies f(n) <= f(m). Suppose for some m, f(m) ≥ m+1, then f(f(m)) ≥ f(m+1). Contradiction. Hence f(m) ≤ m for all m. But since f(1) ≥1 and f(m) > f(m−1) > ... > f(1), we also have f(m) ≥ m. Hence f(m) = m for all m.

IMO 1978

A1

We require $1978^m(1978^{n-m} - 1)$ to be a multiple of 1000=8·125. So we must have 8 divides 1978^m, and hence m ≥ 3, and 125 divides $1978^{n-m} - 1$.

By Euler's theorem, $1978^{\varphi(125)} = 1$ (mod 125). φ(125) = 125 − 25 = 100, so $1978^{100} = 1$ (mod 125). Hence the smallest r such that $1978^r = 1$ (mod 125) must be a divisor of 100 (because if it was not, then the remainder on dividing it into 100 would give a smaller r). That leaves 9 possibilities to check: 1, 2, 4, 5, 10, 20, 25, 50, 100. To reduce the work we quickly find that the smallest s such that $1978^s = 1 \pmod 5$ is 4 and hence r must be a multiple of 4. That leaves 4, 20, 100 to examine.

We find $978^2 = 109 \pmod{125}$, and hence $978^4 = 6 \pmod{125}$. Hence $978^{20} = 6^5 = 36 \cdot 91 = 26 \pmod{125}$. So the smallest r is 100 and hence the solution to the problem is 3, 103.

A2

Suppose ABCD is a rectangle and X any point inside, then $XA^2 + XC^2 = XB^2 + XD^2$. This is most easily proved using coordinates. Take the origin O as the center of the rectangle and take OA to be the vector $\underline{a}$, and OB to be $\underline{b}$. Since it is a rectangle, $|\underline{a}| = |\underline{b}|$. Then OC is $-\underline{a}$ and OD is $-\underline{b}$. Let OX be $\underline{c}$. Then $XA^2 + XC^2 = (\underline{a} - \underline{c})^2 + (\underline{a} + \underline{c})^2 = 2\underline{a}^2 + 2\underline{c}^2 = 2\underline{b}^2 + 2\underline{c}^2 = XB^2 + XD^2$.

Let us fix U. Then the plane k perpendicular to PU through P cuts the sphere in a circle center C. V and W must lie on this circle. Take R so that PVRW is a rectangle. By the result just proved $CR^2 = 2CV^2 - CP^2$. OC is also perpendicular to the plane k. Extend it to X, so that CX = PU. Then extend XU to Y so that YR is perpendicular to k. Now $OY^2 = OX^2 + XY^2 = OX^2 + CR^2 = OX^2 + 2CV^2 - CP^2 = OU^2 - UX^2 + 2CV^2 - CP^2 = OU^2 - CP^2 + 2(OV^2 - OC^2) - CP^2 = 3OU^2 - 2OP^2$. Thus the locus of Y is a sphere.

A3

Let F = {f(1), f(2), f(3), ... }, G = {g(1), g(2), g(3), ... }, N_n = {1, 2, 3, ... , n}. $f(1) \ge 1$, so $f(f(1)) \ge 1$ and hence $g(1) \ge 2$. So 1 is not in G, and hence must be in F. It must be the smallest element of F and so f(1) = 1. Hence g(1) = 2. We can never have two successive integers n and n+1 in G, because if g(m) = n+1, then f(something) = n and so n is in F and G. Contradiction. In particular, 3 must be in F, and so f(2) = 3.

Suppose f(n) = k. Then g(n) = f(k) + 1. So $|N_{f(k)+1} \cap G| = n$. But $|N_{f(k)+1} \cap F| = k$, so n + k = f(k) + 1, or f(k) = n + k − 1. Hence g(n) = n + k. So n + k + 1 must be in F and hence f(k+1) = n + k + 1. This so given the value of f for n we can find it for k and k+1.

Using k+1 each time, we get, successively, f(2) = 3, f(4) = 6, f(7) = 11, f(12) = 19, f(20) = 32, f(33) = 53, f(54) = 87, f(88) = 142, f(143) = 231, f(232) = 375, which is not much help. Trying again with k, we get: f(3) = 4, f(4) = 6, f(6) = 9, f(9) = 14, f(14) = 22, f(22) = 35, f(35) = 56, f(56) = 90, f(90) = 145, f(145) = 234. Still not right, but we can try backing up slightly and using k+1: f(146) = 236. Still not right, we need to back up further: f(91) = 147, f(148) = 239, f(240) = 388.

B1

It is not a good idea to get bogged down in complicated formulae for the various radii. The solution is actually simple.

By symmetry the midpoint, M, is already on the angle bisector of A, so it is sufficient to show it is on the angle bisector of B. Let the angle bisector of A meet the circumcircle again at R. AP is a tangent to the circle touching AB at P, so $\angle PRQ = \angle APQ = \angle ABC$. Now the quadrilateral PBRM is cyclic because the angles PBR, PMR are both 90^o. Hence $\angle PBM = \angle PRM = (\angle PRQ)/2$, so BM does indeed bisect angle B as claimed.

B2

We use the general rearrangement result: given $b_1 \geq b_2 \geq \ldots \geq b_n$, and $c_1 \leq c_2 \leq \ldots \leq c_n$, if $\{a_i\}$ is a permutation of $\{c_i\}$, then $\sum a_i b_i \geq \sum c_i b_i$. To prove it, suppose that $i < j$, but $a_i > a_j$. Then interchanging a_i and a_j does not increase the sum, because $(a_i - a_j)(b_i - b_j) \geq 0$, and hence $a_i b_i + a_j b_j \geq a_j b_i + a_i b_j$. By a series of such interchanges we transform $\{a_i\}$ into $\{c_i\}$ (for example, first swap c_1 into first place, then c_2 into second place and so on).

Hence we do not increase the sum by permuting $\{a_i\}$ so that it is in increasing order. But now we have $a_i > i$, so we do not increase the sum by replacing a_i by i and that gives the sum from 1 to n of 1/k.

B3

The trick is to use differences.

At least 6.329 = 1974, so at least 330 members come from the same country, call it C1. Let their numbers be $a_1 < a_2 < \ldots < a_{330}$. Now take the 329 differences $a_2 - a_1, a_3 - a_1, \ldots , a_{330} - a_1$. If any of them are in C1, then we are home, so suppose they are all in the other five countries.

At least 66 must come from the same country, call it C2. Write the 66 as $b_1 < b_2 < \ldots < b_{66}$. Now form the 65 differences $b_2 - b_1, b_3 - b_1, \ldots , b_{66} - b_1$. If any of them are in C2, then we are home. But each difference equals the difference of two of the original a_is, so if it is in C1 we are also home.

So suppose they are all in the other four countries. At least 17 must come from the same country, call it C3. Write the 17 as $c_1 < c_2 < \ldots < c_{17}$. Now form the 16 differences $c_2 - c_1, c_3 - c_1, \ldots , c_{17} - c_1$. If any of them are in C3, we are home. Each difference equals the difference of two b_is, so if any of them are in C2 we are home. [For example, consider $c_i - c_1$. Suppose $c_i = b_n - b_1$ and $c_1 = b_m - b_1$, then $c_i - c_1 = b_n - b_m$, as claimed.]. Each difference also equals the difference of two a_is, so if any of them are in C1, we are also home. [For example, consider $c_i - c_1$, as before. Suppose $b_n = a_j - a_1$, $b_m = a_k - a_1$, then $c_i - c_1 = b_n - b_m = a_j - a_k$, as claimed.]

So suppose they are all in the other three countries. At least 6 must come from the same country, call it C4. We look at the 5 differences and conclude in the same way that at least 3 must come from C5. Now the 2 differences must both be in C6 and their difference must be in one of the C1, ... , C6 giving us the required sum.

IMO 1979

A1

This is difficult.

The obvious step of combining adjacent terms to give 1/(n(n+1) is unhelpful. The trick is to separate out the negative terms:

1 − 1/2 + 1/3 − 1/4 + ... − 1/1318 + 1/1319 = 1 + 1/2 + 1/3 + ... + 1/1319 − 2(1/2 + 1/4 + ... + 1/1318) = 1/660 + 1/661 + ... + 1/1319.

and to notice that 660 + 1319 = 1979. Combine terms in pairs from the outside:

1/660 + 1/1319 = 1979/(660.1319); 1/661 + 1/1318 = 1979/(661.1318) etc.

There are an even number of terms, so this gives us a sum of terms 1979/m with m not divisible by 1979 (since 1979 is prime and so does not divide any product of smaller numbers). Hence the sum of the 1/m gives a rational number with denominator not divisible by 1979 and we are done.

A2

We show first that the A_i are all the same color. If not then, there is a vertex, call it A_1, with edges A_1A_2, A_1A_5 of opposite color. Now consider the five edges A_1B_i. At least three of them must be the same color. Suppose it is green and that A_1A_2 is also green. Take the three edges to be A_1B_i, A_1B_j, A_1B_k. Then considering the triangles $A_1A_2B_i$, $A_1A_2B_j$, $A_1A_2B_k$, the three edges A_2B_i, A_2B_j, A_2B_k must all be red. Two of B_i, B_j, B_k must be adjacent, but if the resulting edge is red then we have an all red triangle with A_2, whilst if it is green we have an all green triangle with A_1. Contradiction. So the A_i are all the same color. Similarly, the B_i are all the same color. It remains to show that they are the same color. Suppose otherwise, so that the A_i are green and the B_i are red.

Now we argue as before that 3 of the 5 edges A_1B_i must be the same color. If it is red, then as before 2 of the 3 B_i must be adjacent and that gives an all red triangle with A_1. So 3 of the 5 edges A_1B_i must be green. Similarly, 3 of the 5 edges A_2B_i must be green. But there must be a B_i featuring in both sets and it forms an all green triangle with A_1 and A_2. Contradiction. So the A_i and the B_i are all the same color.

A3

Let the circles have centers O, O' and let the moving points by X, X. Let P be the reflection of A in the perpendicular bisector of OO'. We show that triangles POX, X'O'P are congruent. We have OX = OA (pts on circle) = O'P (reflection). Also OP = O'A (reflection) = O'X' (pts on circle). Also ∠AOX = ∠AO'X' (X and X' circle at same rate), and ∠AOP = ∠AO'P (reflection), so ∠POX = ∠PO'X'. So the triangles are congruent. Hence PX = PX'.

Another approach is to show that XX' passes through the other point of intersection of the two circles, but that involves looking at many different cases depending on the relative positions of the moving points.

B1

Consider the points R on a circle center P. Let X be the foot of the perpendicular from Q to k. Assume P is distinct from X, then we minimise QR (and hence maximise (QP + PR)/QR) for points R on the circle by taking R on the line PX. Moreover, R must lie on the same side of P as X. Hence if we allow R to vary over k, the points maximising (QP + PR)/QR must lie on the ray PX. Take S on the line PX on the opposite side of P from X so that PS = PQ. Then for points R on the ray PX we have (QP + PR)/QR = SR/QR = sin RQS/sin QSR. But sin QSR is fixed for points on the ray, so we maximise the ratio by taking ∠RQS = 90^o. Thus there is a single point maximising the ratio.

If P = X, then we still require ∠RQS = 90^o, but R is no longer restricted to a line, so it can be anywhere on a circle center P.

B2

Take a^2 x 1st equ – 2a x 2nd equ + 3rd equ. The rhs is 0. On the lhs the coefficient of x_n is $a^2n - 2an^3 + n^5 = n(a - n^2)^2$. So the lhs is a sum of non-negative terms. Hence each term must be zero separately, so for each n either $x_n = 0$ or $a = n^2$. So there are just 5 solutions, corresponding to a = 1, 4, 9, 16, 25. We can check that each of these gives a solution. [For $a = n^2$, $x_n = n$ and the other x_i are zero.]

B3

Each jump changes the parity of the shortest distance to E. The parity is initially even, so an odd number of jumps cannot end at E. Hence $a_{2n-1} = 0$. We derive a recurrence relation for a_{2n}. This is not easy to do directly, so we introduce b_n which is the number of paths length n from C to E. Then we have immediately:

$a_{2n} = 2a_{2n-2} + 2b_{2n-2}$ for $n > 1$
$b_{2n} = 2b_{2n-2} + a_{2n-2}$ for $n > 1$

Hence, using the first equation: $a_{2n} - 2a_{2n-2} = 2a_{2n-2} - 4a_{2n-4} + 2b_{2n-2} - 4b_{2n-4}$ for $n > 2$. Using the second equation, this leads to: $a_{2n} = 4a_{2n-2} - 2a_{2n-4}$ for $n > 2$. This is a linear recurrence relation with the general solution: $a_{2n} = a(2 + \sqrt{2})^{n-1} + b(2 - \sqrt{2})^{n-1}$. But we easily see directly that $a_4 = 2$, $a_6 = 8$ and we can now solve for the coefficients to get the solution given.

IMO 1981

A1

We have PD.BC + PE.CA + PF.AB = 2 area of triangle. Now use Cauchy's inequality with $x_1 = \sqrt{(PD \cdot BC)}$, $x_2 = \sqrt{(PE \cdot CA)}$, $x_3 = \sqrt{(PF \cdot AB)}$, and $y_1 = \sqrt{(BC/PD)}$, $y_2 = \sqrt{(CA/PE)}$, $y_3 = \sqrt{(AB/PF)}$. We get that $(BC + CA + AB)^2 < 2$ x area of triangle x (BC/PD + CA/PE + AB/PF) with equality only if x_i/y_i = const, ie PD = PE = PF. So the unique minimum position for P is the incenter.

A2

Denote the binomial coefficient n!/(r!(n−r)!) by nCr.

Evidently nCr F(n,r) = 1 (n−1)C(r−1) + 2 (n−2)C(r−1) + ... + (n−r+1) (r−1)C(r−1). [The first term denotes the contribution from subsets with smallest element 1, the second term smallest element 2 and so on.]

Let the rhs be g(n,r). Then, using the relation (n−i)C(r−1) − (n−i−1)C(r−2) = (n−i−1)C(r−1), we find that g(n,r) − g(n−1,r−1) = g(n−1,r), and we can extend this relation to r=1 by taking g(n,0) = n+1 = (n+1)C1. But g(n,1) = 1 + 2 + ... + n = n(n+1)/2 = (n+1)C2. So it now follows by an easy induction that g(n,r) = (n+1)C(r+1) = nCr (n+1)/(r+1). Hence F(n,r) = (n+1)/(r+1).

A more elegant solution by Oliver Nash is as follows

Let k be the smallest element. We want to evaluate g(n, r) = $\sum_{k=1 \text{ to } n-r+1}$ k   (n−k)C(r−1). Consider the subsets with r+1 elements taken from 1, 2, 3, ... , n+1. Suppose k+1 is the second smallest element. Then there are k   (n−k)C(r−1) possible subsets. So g(n, r) = (n+1)C(r+1). Hence F(n, r) = (n+1)C(r+1) / nCr = (n+1)/(r+1), as required.

A3

Experimenting with small values suggests that the solutions of $n^2 - mn - m^2 = 1$ or -1 are successive Fibonacci numbers. So suppose $n > m$ is a solution. This suggests trying m+n, n: $(m+n)^2 - (m+n)n - n^2 = m^2 + mn - n^2 = -(n^2 - mn - m^2) = 1$ or -1. So if $n > m$ is a solution, then m+n, n is another solution.

Running this forward from 2,1 gives 3,2; 5,3; 8,5; 13,8; 21,13; 34,21; 55,34; 89,55; 144,89; 233,144; 377,233; 610,377; 987,610; 1597,987; 2584,1597.

But how do we know that there are no other solutions? The trick is to run the recurrence the other way. For suppose n > m is a solution, then try m, n−m: $m^2 - m(n-m) - (n-m)^2 = m^2 + mn - n^2 = -(n^2 - mn - m^2) = 1$ or −1, so that also satisfies the equation. Also if m > 1, then m > n−m (for if not, then n >= 2m, so $n(n - m) >= 2m^2$, so $n^2 - nm - m^2 >= m^2 > 1$). So given a solution n > m with m > 1, we have a smaller solution m > n−m. This process must eventually terminate, so it must finish at a solution n, 1 with n > 1. But the only such solution is 2, 1. Hence the starting solution must have been in the forward sequence from 2, 1.

Hence the solution to the problem stated is $1597^2 + 987^2$.

B1

(a) n = 3 is not possible. For suppose x was the largest number in the set. Then x cannot be divisible by 3 or any larger prime, so it must be a power of 2. But it cannot be a power of 2, because $2^m - 1$ is odd and $2^m - 2$ is not a positive integer divisible by 2^m.

For k ≥ 2, the set 2k−1, 2k , ... , 4k−2 gives n = 2k. For k ≥ 3, so does the set 2k−5, 2k−4, ... , 4k−6. For k ≥ 2, the set 2k−2, 2k−3, ... , 4k−2 gives n = 2k+1. For k ≥ 4 so does the set 2k−6,2k−5, ... , 4k−6. So we have at least one set for every n ≥ 4, which answers (a).

(b) We also have at least two sets for every n ≥ 4 except possibly n = 4, 5, 7. For 5 we may take as a second set: 8, 9, 10, 11, 12, and for 7 we may take 6, 7, 8, 9 ,10, 11, 12. That leaves n = 4. Suppose x is the largest number in a set with n =4. x cannot be divisible by 5 or any larger prime, because x−1, x−2, x−3 will not be. Moreover, x cannot be divisible by 4, because then x−1 and x−3 will be odd, and x−2 only divisible by 2 (not 4). Similarly, it cannot be divisible by 9. So the only possibilities are 1, 2, 3, 6. But we also require x ≥ 4, which eliminates the first three. So the only solution for n = 4 is the one we have already found: 3, 4, 5, 6.

B2

Let the triangle be ABC. Let the center of the circle touching AB and AC be D, the center of the circle touching AB and BC be E, and the center of the circle touching AC and BC be F. Because the circles center D and E have the same radius the perpendiculars from D and E to AB have the same length, so DE is parallel to AB. Similarly EF is parallel to BC and FD is parallel to CA. Hence DEF is similar and similarly oriented to ABC. Moreover D must lie on the angle bisector of A since the circle center D touches AB and AC. Similarly E lies on the angle bisector of B and F lies on the angle bisector of C. Hence the incenter I of ABC is also the incenter of DEF and acts as a center of symmetry so that corresponding points P of ABC and P' of DEF lie on a line through I with PI/P'I having a fixed ratio. But OD = OE = OF since the three circles have equal radii, so O is the circumcenter of DEF. Hence it lies on a line with I and the circumcenter of ABC.

B3

f(1,n) = f(0,f(1,n−1)) = 1 + f(1,n−1). So f(1,n) = n + f(1,0) = n + f(0,1) = n + 2.

f(2,n) = f(1,f(2,n−1)) = f(2,n−1) + 2. So f(2,n) = 2n + f(2,0) = 2n + f(1,1) = 2n + 3.

f(3,n) = f(2,f(3,n−1)) = 2f(3,n−1) + 3. Let u_n = f(3,n) + 3, then $u_n = 2u_{n-1}$. Also u_0 = f(3,0) + 3 = f(2,1) + 3 = 8. So $u_n = 2^{n+3}$, and f(3,n) = $2^{n+3} - 3$.

f(4,n) = f(3,f(4,n−1)) = $2^{f(4,n-1)+3} - 3$. f(4,0) = f(3,1) = $2^4 - 3$ = 13. We calculate two more terms to see the pattern: f(4,1) = $2^{24} - 3$, f(4,2) = $2^{224} - 3$. In fact it looks neater if we replace 4 by 2^2, so that f(4,n) is a tower of n+3 2s less 3.

IMO 1982

A1

We show that f(n) = [n/3] for n <= 9999, where [ ] denotes the integral part.

We show first that f(3) = 1. f(1) must be 0, otherwise f(2) − f(1) − f(1) would be negative. Hence f(3) = f(2) + f(1) + 0 or 1 = 0 or 1. But we are told f(3) > 0, so f(3) = 1. It follows by induction that f(3n) ≥ n. For f(3n+3) = f(3) + f(3n) + 0 or 1 = f(3n) + 1 or 2. Moreover if we ever get f(3n) > n, then the same argument shows that f(3m) > m for all m > n. But f(3.3333) = 3333, so f(3n) = n for all n <= 3333.

Now f(3n+1) = f(3n) + f(1) + 0 or 1 = n or n + 1. But 3n+1 = f(9n+3) ≥ f(6n+2) + f(3n+1) ≥ 3f(3n+1), so f(3n+1) < n+1. Hence f(3n+1) = n. Similarly, f(3n+2) = n. In particular f(1982) = 660.

A2

Let B_i be the point of intersection of the interior angle bisector of the angle at A_i with the opposite side. The first step is to figure out which side of B_i T_i lies. Let A_1 be the largest angle, followed by A_2. Then T_2 lies between A_1 and B_2, T_3 lies between A_1 and B_3, and T_1 lies between A_2 and B_1. For □$OB_2A_1 = 180^o - A_1 - A_2/2 = A_3 + A_2/2$. But $A_3 + A_2/2 < A_1 + A_2/2$ and their sum is 180^o, so $A_3 + A_2/2 < 90^o$. Hence T_2 lies between A_1 and B_2. Similarly for the others.

Let O be the center of the incircle. Then $\angle T_1OS_2 = \angle T_1OT_2 - 2\angle T_2OB_2 = 180^o - A_3 - 2(90^o - \angle OB_2T_2) = 2(A_3 + A_2/2) - A_3 = A_2 + A_3$. A similar argument shows $\angle T_1OS_3 = A_2 + A_3$. Hence S_2S_3 is parallel to A_2A_3.

Now $\angle T_3OS_2 = 360^o - \angle T_3OT_1 - \angle T_1OS_2 = 360^o - (180^o - A_2) - (A_2 + A_3) = 180^o - A_3 = A_1 + A_2$. $\angle T_3OS_1 = \angle T_3OT_1 + 2\angle T_1OB_1 = (180^o - A_2) + 2(90^o - \angle OB_1T_1) = 360^o - A_2 - 2(A_3 + A_1/2) = 2(A_1 + A_2 + A_3) - A_2 - 2A_3 - A_1 = A_1 + A_2 = \angle T_3OS_2$. So S_1S_2 is parallel to A_1A_2. Similarly we can show that S_1S_3 is parallel to A_1A_3.

So $S_1S_2S_3$ is similar to $A_1A_2A_3$ and turned through 180^o. But $M_1M_2M_3$ is also similar to $A_1A_2A_3$ and turned through 180^o. So $S_1S_2S_3$ and $M_1M_2M_3$ are similar and similarly oriented. Hence the lines through corresponding vertices are concurrent.

A3

(a) It is sufficient to show that the sum of the (infinite) sequence is at least 4. Let k be the greatest lower bound of the limits of all such sequences. Clearly $k \geq 1$. Given any $\varepsilon > 0$, we can find a sequence $\{x_n\}$ with sum less than $k + \varepsilon$. But we may write the sum as:

$x_0^2/x_1 + x_1((x_1/x_1)^2/(x_2/x_1) + (x_2/x_1)^2/(x_3/x_1) + \dots + (x_n/x_1)^2/(x_{n+1}/x_1) + \dots)$.

The term in brackets is another sum of the same type, so it is at least k. Hence $k + \varepsilon > 1/x_1 + x_1k$. This holds for all $\varepsilon > 0$, and so $k \geq 1/x_1 + x_1k$. But $1/x_1 + x_1k \geq 2\sqrt{k}$, so $k \geq 4$.

(b) Let $x_n = 1/2^n$. Then $x_0^2/x_1 + x_1^2/x_2 + \dots + x_{n-1}^2/x_n = 2 + 1 + 1/2 + \dots + 1/2^{n-2} = 4 - 1/2^{n-2} < 4$.

B1

If x, y is a solution then so is $y-x, -x$. Hence also $-y, x-y$. If the first two are the same, then $y = -x$, and $x = y-x = -2x$, so $x = y = 0$, which is impossible, since $n > 0$. Similarly, if any other pair are the same.

$2891 = 2 \pmod 9$ and there is no solution to $x^3 - 3xy^2 + y^3 = 2 \pmod 9$. The two cubes are each -1, 0 or 1, and the other term is 0, 3 or 6, so the only solution is to have the cubes congruent to 1 and -1 and the other term congruent to 0. But the other term cannot be congruent to 0, unless one of x, y is a multiple of 3, in which case its cube is congruent to 0, not 1 or -1.

B2

For an inelegant solution one can use coordinates. The advantage of this type of approach is that it is quick and guaranteed to work! Take A as $(0,\sqrt{3})$, B as

$(1,\sqrt{3})$, C as $(3/2,\sqrt{3}/2$, D as (1,0). Take the point X, coordinates (x,0), on ED. We find where the line BX cuts AC and CE. The general point on BX is $(k + (1-k)x,k\sqrt{3})$. If this is also the point M with AM/AC = r then we have: $k + (1-k)x = 3r/2$, $k\sqrt{3} = (1-r)\sqrt{3} + r\sqrt{3}/2$. Hence $k = 1 - r/2$, $r = 2/(4-x)$. Similarly, if it is the point N with CN/CE = r, then $k + (1-k)x = 3(1-r)/2$, $k\sqrt{3} = (1-r)\sqrt{3}/2$. Hence $k = (1-r)/2$ and $r = (2-x)/(2+x)$. Hence for the ratios to be equal we require $2/(4-x) = (2-x)/(2+x)$, so $x^2 - 8x + 4 = 0$. We also have $x < 1$, so $x = 4 - \sqrt{12}$. This gives $r = 1/\sqrt{3}$.

A more elegant solution uses the ratio theorem for the triangle EBC. We have CM/MX XB/BE EN/NC = −1. Hence $(1-r)/(r - 1/2)\ (-1/4)\ (1-r)/r = -1$. So $r = 1/\sqrt{3}$.

B3

Let the square be A'B'C'D'. The idea is to find points of L close to a particular point of A'D' but either side of an excursion to B'.

We say L approaches a point P' on the boundary of the square if there is a point P on L with $PP' \le 1/2$. We say L approaches P' before Q' if there is a point P on L which is nearer to A_0 (the starting point of L) than any point Q with $QQ' \le 1/2$.

Let A' be the first vertex of the square approached by L. L must subsequently approach both B' and D'. Suppose it approaches B' first. Let B be the first point on L with $BB' \le 1/2$. We can now divide L into two parts L_1, the path from A_0 to B, and L_2, the path from B to A_n.

Take X' to be the point on A'D' closest to D' which is approached by L_1. Let X be the corresponding point on L_1. Now every point on X'D' must be approached by L_2 (and X'D' is non-empty, because we know that D' is approached by L but not by L_1). So by compactness X' itself must be approached by L_2. Take Y to be the corresponding point on L_2. $XY \le XX' + X'Y \le 1/2 + 1/2 = 1$. Also $BB' \le 1/2$, so $XB \ge X'B' - XX' - BB' \ge X'B' - 1 \ge A'B' - 1 = 99$. Similarly $YB \ge 99$, so the path $XY \ge 198$.

IMO 1984

A1

$(1 - 2x)(1 - 2y)(1 - 2z) = 1 - 2(x + y + z) + 4(yz + zx + xy) - 8xyz = 4(yz + zx + xy) - 8xyz - 1$. Hence $yz + zx + xy - 2xyz = 1/4\ (1 - 2x)(1 - 2y)(1 - 2z) + 1/4$. By the arithmetic/geometric mean theorem $(1 - 2x)(1 - 2y)(1 - 2z) \le ((1 - 2x + 1 - 2y + 1 - 2z)/3)^3 = 1/27$. So $yz + zx + xy - 2xyz \le 1/4\ 28/27 = 7/27$.

A2

We find that $(a + b)^7 - a^7 - b^7 = 7ab(a + b)(a^2 + ab + b^2)^2$. So we must find a, b such that $a^2 + ab + b^2$ is divisible by 7^3.

At this point I found a = 18, b = 1 by trial and error.

A more systematic argument turns on noticing that $a^2 + ab + b^2 = (a^3 - b^3)/(a - b)$, so we are looking for a, b with $a^3 = b^3 \pmod{7^3}$. We now remember that $a^{\varphi(m)} = 1 \pmod{m}$. But $\varphi(7^3) = 2\cdot3\cdot49$, so $a^3 = 1 \pmod{343}$ if $a = n^{98}$. We find $2^{98} = 18$ (343), which gives the solution 18, 1.

This approach does not give a flood of solutions. $n^{98} = 0, 1, 18$, or 324. So the only solutions we get are 1, 18; 18, 324; 1, 324.

A3

Suppose the result is false. Let C^1 be any circle center O. Then the locus of points X such that $C(X) = C_1$ is a spiral from O to the point of intersection of OA and C_1. Every point of this spiral must be a different color from all points of the circle C_1. Hence every circle center O with radius smaller than C_1 must include a point of different color to those on C_1. Suppose there are n colors. Then by taking successively smaller circles $C_2, C_3, \ldots, C_{n+1}$ we reach a contradiction, since each circle includes a point of different color to those on any of the larger circles.

B1

If AB and CD are parallel, then AB is tangent to the circle on diameter CD if and only if AB = CD and hence if and only if ABCD is a parallelogram. So the result is true.

Suppose then that AB and DC meet at O. Let M be the midpoint of AB and N the midpoint of CD. Let S be the foot of the perpendicular from N to AB, and T the foot of the perpendicular fromM to CD. We are given that MT = MA. OMT, ONS are similar, so OM/MT = ON/NS and hence OB/OA = (ON − NS)/(ON + NS). So AB is tangent to the circle on diameter CD if and only if OB/OA = OC/OD which is the condition for BC to be parallel to AD.

B2

Given any diagonal AX, let B be the next vertex counterclockwise from A, and Y the next vertex counterclockwise from X. Then the diagonals AX and BY intersect at K. AK + KB > AB and XK + KY > XY, so AX + BY > AB + XY. Keeping A fixed and summing over X gives n − 3 cases. So if we then sum over A we get every diagonal appearing four times on the lhs and every side appearing 2(n−3) times on the rhs, giving 4d > 2(n−3)p.

Denote the vertices as $A_0, \dots, A_{n-1}$ and take subscripts mod n. The ends of a diagonal AX are connected by r sides and n−r sides. The idea of the upper limit is that its length is less than the sum of the shorter number of sides. Evaluating it is slightly awkward.

We consider n odd and n even separately. Let n = 2m+1. For the diagonal A_iA_{i+r} with $r \le m$, we have $A_iA_{i+r} \le A_iA_{i+2} + \dots + A_iA_{i+r}$. Summing from r = 2 to m gives for the rhs $(m-1)A_iA_{i+1} + (m-1)A_{i+1}A_{i+2} + (m-2)A_{i+2}A_{i+3} + (m-3)A_{i+3}A_{i+4} + \dots + 1.A_{i+m-1}A_{i+m}$. Now summing over i gives d for the lhs and $p((m-1) + (1 + 2 + \dots + m-1)) = p((m^2 + m - 2)/2)$ for the rhs. So we get $2d/p \le m^2 + m - 2 = [n/2]\ [(n+1)/2] - 2$.

Let n = 2m. As before we have $A_iA_{i+r} <= A_iA_{i+2} + \dots + A_iA_{i+r}$ for $2 \le r \le m-1$. We may also take $A_iA_{i+m} \le p/2$. Summing as in the even case we get $2d/p = m^2 - 2 = [n/2]\ [(n+1)/2] - 2$.

B3

a < c, so a(d − c) < c (d − c) and hence bc − ac < c(d − c). So b − a < d − c, or a + d > b + c, so k > m.

bc = ad, so $b(2^m - b) = a(2^k - a)$. Hence $b^2 - a^2 = 2^m(b - 2^{k-m}a)$. But $b^2 - a^2 = (b + a)(b - a)$, and (b + a) and (b − a) cannot both be divisible by 4 (since a and b are odd), so 2^{m-1} must divide b + a or b − a. But if it divides b − a, then $b - a \ge 2^{m-1}$, so b and c > 2^{m-1} and b + c > 2^m. Contradiction. Hence 2^{m-1} divides b + a. If $b + a \ge 2^m = b + c$, then $a \ge c$. Contradiction. Hence $b + a = 2^{m-1}$.

So we have $b = 2^{m-1} - a$, $c = 2^{m-1} + a$, $d = 2^k - a$. Now using bc = ad gives: $2^ka = 2^{2m-2}$. But a is odd, so a = 1.

IMO 1985

A1

Let the circle touch AD, CD, BC at L, M, N respectively. Take X on the line AD on the same side of A as D, so that AX = AO, where O is the center of the circle. Now the triangles OLX and OMC are congruent: OL = OM = radius of circle, □OLX = □OMC = 90°, and □OXL = 90° − A/2 = (180° − A)/2 = C/2 (since ABCD is cyclic) = □OCM. Hence LX = MC. So OA = AL + MC. Similarly, OB = BN + MD. But MC = CN and MD = DL (tangents have equal length), so AB = OA + OB = AL + LD + CN + NB = AD + BC.

A2

n and k are relatively prime, so 0, k, 2k, ... , (n−1)k form a complete set of residues mod n. So k, 2k, ... , (n−1)k are congruent to the numbers 1, 2, ... , n−1 in some order. Suppose ik is congruent to r and (i+1)k is congruent to s. Then either s = r + k, or s = r + k − n. If s = r + k, then we have immediately that r = s

– k and s have the same color. If s = r + k – n, then r = n – (k – s), so r has the same color as k – s, and k – s has the same color as s. So in any case r and s have the same color. By giving i values from 1 to n–2 this establishes that all the numbers have the same color.

A3

If i is a power of 2, then all coefficients of Q_i are even except the first and last. [There are various ways to prove this. Let iCr denote the rth coefficient, so iCr = i!/(r!(i–r)!). Suppose 0 < r < i. Then iCr = i–1Cr–1 i/r, but i–1Cr–1 is an integer and i is divisible by a higher power of 2 than r, hence iCr is even.]

Let $Q = Q_{i1} + \dots + Q_{in}$. We use induction on i_n. If $i_n = 1$, then we must have n = 2, $i_1 = 0$, and $i_2 = 1$, so Q = 2 + x, which has the same number of odd coefficients as $Q_{i1} = 1$. So suppose it is true for smaller values of i_n. Take m a power of 2 so that $m \le i_n < 2m$. We consider two cases $i_1 \ge m$ and $i_1 < m$.

Consider first $i_1 \ge m$. Then $Q_{i1} = (1 + x)^m A$, $Q = (1 + x)^m B$, where A and B have degree less than m. Moreover, A and B are of the same form as Q_{i1} and Q, (all the i_js are reduced by m, so we have $o(A) \le o(B)$ by induction. Also $o(Q_{i1}) = o((1 + x)^m A) = o(A + x^m A) = 2o(A) \le 2o(B) = o(B + x^m B) = o((1 + x)^m B) = o(Q)$, which establishes the result for i_n.

It remains to consider the case $i_1 < m$. Take r so that $i_r < m$, $i_{r+1} > m$. Set $A = Q_{i1} + \dots + Q_{ir}$, $(1 + x)^m B = Q_{ir+1} + \dots + Q_{in}$, so that A and B have degree < m. Then $o(Q) = o(A + (1 + x)^m B) = o(A + B + x^m B) = o(A + B) + o(B)$. Now o(A – B) + o(B) >= o(A – B + B) = o(A), because a coefficient of A is only odd if just one of the corresponding coefficients of A – B and B is odd. But o(A – B) = o(A + B), because corresponding coefficients of A – B and A + B are either equal or of the same parity. Hence $o(A + B) + o(B) \ge o(A)$. But $o(A) \ge o(Q_{ii})$ by induction. So we have established the result for i_n.

B1

Suppose we have a set of at least 3.2^n+1 numbers whose prime divisors are all taken from a set of n. So each number can be written as $p_1^{r_1} \dots p_n^{r_n}$ for some non-negative integers r_i, where p_i is the set of prime factors common to all the numbers. We classify each r_i as even or odd. That gives 2^n possibilities. But there are more than $2^n + 1$ numbers, so two numbers have the same classification and hence their product is a square. Remove those two and look at the remaining numbers. There are still more than $2^n + 1$, so we can find another pair. We may repeat to find $2^n + 1$ pairs with a square product. [After removing 2^n pairs, there are still $2^n + 1$ numbers left, which is just enough to find the final pair.] But we may now classify these pairs according to whether each exponent in the square root of their product is odd or even. We must find two pairs with the same classification. The product of these four numbers is now a fourth power.

Applying this to the case given, there are 9 primes less than or equal to 23 (2, 3, 5, 7, 11, 13, 17, 19, 23), so we need at least 3.512 + 1 = 1537 numbers for the argument to work (and we have 1985).

The key is to find the 4th power in two stages, by first finding lots of squares. If we try to go directly to a 4th power, this type of argument does not work (we certainly need more than 5 numbers to be sure of finding four which sum to 0 mod 4, and 5^9 is far too big).

B2

The three radical axes of the three circles taken in pairs, BM, NK and AC are concurrent. Let X be the point of intersection. [They cannot all be parallel or B and M would coincide.] The first step is to show that XMNC is cyclic. The argument depends slightly on how the points are arranged. We may have: $\angle XMN = 180^o - \angle BMN = \angle BKN = 180^o - \angle AKN = \angle ACN = 180^o - \angle XCN$, or we may have $\angle XMN = 180^o - \angle BMN = 180^o - \angle BKN = \angle AKN = 180^o - \angle ACN = 180^o - \angle XCN$.

Now $XM.XB = XK.XN = XO^2 - ON^2$. $BM\cdot BX = BN\cdot BC = BO^2 - ON^2$, so $XM\cdot XB - BM\cdot BX = XO^2 - BO^2$. But $XM\cdot XB - BM\cdot BX = XB(XM - BM) = (XM + BM)(XM - BM) = XM^2 - BM^2$. So $XO^2 - BO^2 = XM^2 - BM^2$. Hence OM is perpendicular to XB, or $\angle OMB = 90^o$.

B3

Define $S_0(x) = x$, $S_n(x) = S_{n-1}(x)\ (S_{n-1}(x) + 1/n)$. The motivation for this is that $x_n = S_{n-1}(x_1)$.

$S_n(0) = 0$ and $S_n(1) > 1$ for all $n > 1$. Also $S_n(x)$ has non-negative coefficients, so it is strictly increasing in the range [0,1]. Hence we can find (unique) solutions a_n, b_n to $S_n(a_n) = 1 - 1/n$, $S_n(b_n) = 1$.

$S_{n+1}(a_n) = S_n(a_n)\ (S_n(a_n) + 1/n) = 1 - 1/n > 1 - 1/(n+1)$, so $a_n < a_{n+1}$. Similarly, $S_{n+1}(b_n) = S_n(b_n)\ (S_n(b_n) + 1/n) = 1 + 1/n > 1$, so $b_n > b_{n+1}$. Thus a_n is an increasing sequence and b_n is a decreasing sequence with all a_n less than all b_n. So we can certainly find at least one point x_1 which is greater than all the a_n and less than all the b_n. Hence $1 - 1/n < S_n(x_1) < 1$ for all n. But $S_n(x_1) = x_{n+1}$. So $x_{n+1} < 1$ for all n. Also $x_n > 1 - 1/n$ implies that $x_{n+1} = x_n(x_n + 1/n) > x_n$. Finally, we obviously have $x_n > 0$. So the resulting series x_n satisfies all the required conditions.

It remains to consider uniqueness. Suppose that there is an x_1 satisfying the conditions given. Then we must have $S_n(x_1)$ lying in the range $1 - 1/n$, 1 for all n. [The lower limit follows from $x_{n+1} = x_n(x_n + 1/n)$.] Hence we must have $a_n < x_1 < b_n$ for all n. We show uniqueness by showing that $b_n - a_n$ tends to zero as n tends to infinity. Since all the coefficients of $S_n(x)$ are non-negative, it is has increasing derivative. $S_n(0) = 0$ and $S_n(b_n) = 1$, so for any x in the range 0, b_n we have $S_n(x) \le x/b_n$. In particular, $1 - 1/n < a_n/b_n$. Hence $b_n - a_n \le b_n - b_n(1 - 1/n) = b_n/n < 1/n$, which tends to zero.

IMO 1986

A1

Consider residues mod 16. A perfect square must be 0, 1, 4 or 9 (mod 16). d must be 1, 5, 9, or 13 for 2d – 1 to have one of these values. However, if d is 1 or 13, then 13d – 1 is not one of these values. If d is 5 or 9, then 5d – 1 is not one of these values. So we cannot have all three of 2d – 1, 5d – 1, 13d – 1 perfect squares.

Alternative solution from Marco Dalai

Suppose 2d–1, 5d–1, 13d–1 are all squares. Squares mod 4 must be 0 or 1, considering 2d–1, so d must be odd. Put d = 2k+1. Then $10k+4 = b^2$. So b must be even, so k must be even. Put k = 2h, then 5k+1 is a square. Similarly, 52h+12 is a square, so 13h+3 is a square. Hence (13h+3)–(5h+1) = 8h+2 is a difference of two squares, which is impossible (a difference of two squares must be 0, 1, or 3 mod 4).

A2

The product of three successive rotations about the three vertices of a triangle must be a translation (see below). But that means that P_{1986} (which is the result of 662 such operations, since 1986 = 3 x 662) can only be P_0 if it is the identity, for a translation by a non-zero amount would keep moving the point further away. It is now easy to show that it can only be the identity if the triangle is equilateral. Take a circle center A_1, radius A_1A_2 and take P on the circle so that a 120^o clockwise rotation about A_1 brings P to A_2. Take a circle center A_3, radius A_3A_2 and take Q on the circle so that a 120^o clockwise rotation about A_3 takes A_2 to Q. Then successive 120^o clockwise rotations about A_1, A_2, A_3 take P to Q. So if these three are equivalent to the identity we must have P = Q. Hence □$A_1A_2A_3$ = □A_1A_2P + □A_3A_2P = $30^o + 30^o = 60^o$. Also $A_2P = 2A_1A_2\cos 30^o$ and = $2A_2A_3\cos 30^o$. Hence $A_1A_2 = A_2A_3$. So $A_1A_2A_2$ is equilateral. Note in passing that it is not sufficient for the triangle to be equilateral. We also have to take the rotations in the right order. If we move around the vertices the opposite way, then we get a net translation.

It remains to show that the three rotations give a translation. Define rectangular coordinates (x, y) by taking A_1 to be the origin and A_2 to be (a, b). Let A_3 be (c, d). A clockwise rotation through 120 degrees about the origin takes (x, y) to $(-x/2 + y\sqrt{3}/2, -x\sqrt{3}/2 - y/2)$. A clockwise rotation through 120 degrees about some other point (e, f) is obtained by subtracting (e, f) to get (x – e, y – f), the coordinates relative to (e, f), then rotating, then adding (e, f) to get the coordinates relative to (0, 0). Thus after the three rotations we will end up with a linear combination of x's, y's, a's, b's, c's and d's for each coordinate. But the linear combination of x's and y's must be just x for the x-coordinate and y for the y-coordinate, since three successive 120 degree rotations about the same point is the identity. Hence we end up with simply (x + constant, y + constant), in other words, a translation.

[Of course, there is nothing to stop you actually carrying out the computation. It makes things slightly easier to take the triangle to be (0, 0), (1, 0), (a, b). The net result turns out to be (x, y) goes to $(x + 3a/2 - b\sqrt{3}/2,\ y - \sqrt{3} + a\sqrt{3}/2 + 3b/2)$. For this to be the identity requires $a = 1/2$, $b = \sqrt{3}/2$. So the third vertex must make the triangle equilateral (and it must be on the correct side of the line joining the other two). This approach avoids the need for the argument in the first paragraph above, but is rather harder work.]

A3

Let S be the sum of the absolute value of each set of adjacent vertices, so if the integers are a, b, c, d, e, then $S = |a| + |b| + |c| + |d| + |e| + |a + b| + |b + c| + |c + d| + |d + e| + |e + a| + |a + b + c| + |b + c + d| + |c + d + e| + |d + e + a| + |e + a + b| + |a + b + c + d| + |b + c + d + e| + |c + d + e + a| + |d + e + a + b| + |e + a + b + c| + |a + b + c + d + e|$. Then the operation reduces S, but S is a greater than zero, so the process must terminate in a finite number of steps. So see that S is reduced, we can simply write out all the terms. Suppose the integers are a, b, c, d, e before the operation, and a+b, −b, b+c, d, e after it. We find that we mostly get the same terms before and after (although not in the same order), so that the sum S' after the operation is $S - |a + c + d + e| + |a + 2b + c + d + e|$. Certainly $a + c + d + e > a + 2b + c + d + e$ since b is negative, and $a + c + d + e > -(a + 2b + c + d + e)$ because $a + b + c + d + e > 0$.

S is not the only expression we can use. If we take $T = (a - c)^2 + (b - d)^2 + (c - e)^2 + (d - a)^2 + (e - b)^2$*, then after replacing a, b, c by a+b, −b, b+c, we get* $T' = T + 2b(a + b + c + d + e) < T$*. Thanks to Demetres Chrisofides for T*

B1

Take AB = 2 and let M be the midpoint of AB. Take coordinates with origin at A, x-axis as AB and y-axis directed inside the n-gon. Let Z move along AB from B towards A. Let □YZA be t. Let the coordinates of X be (x, y). □YZX = $\pi/2 - \pi/n$, so $XZ = 1/\sin \pi/n$ and $y = XZ \sin(t + \pi/2 - \pi/n) = \sin t + \cot \pi/n \cos t$.

$BY \sin 2\pi/n = YZ \sin t = 2 \sin t$. $MX = \cot \pi/n$. So $x = MY \cos t - BY \cos 2\pi/n + MX \sin t = \cos t + (\cot \pi/n - 2 \cot 2\pi/n) \sin t = \cos t + \tan \pi/n \sin t = y \tan \pi/n$. Thus the locus of X is a star formed of n lines segments emanating from O. X moves out from O to the tip of a line segement and then back to O, then out along the next segment and so on. $x^2 + y^2 = (1/\sin^2\pi/n + 1/\cos^2\pi/n) \cos^2(t + \pi/n)$. Thus the length of each segment is $(1 - \cos \pi/n)/(\sin \pi/n \cos \pi/n)$.

B2

$f(x+2) = f(xf(2)) f(2) = 0$. So $f(x) = 0$ for all $x \geq 2$.

$f(y) f((2-y)f(y)) = f(2) = 0$. So if $y < 2$, then $f((2-y) f(y)) = 0$ and hence $(2 - y) f(y) \geq 2$, or $f(y) \geq 2/(2 - y)$.

Suppose that for some y_0 we have $f(y_0) > 2/(2 - y_0)$, then we can find $y_1 > y_0$ (and $y_1 < 2$) so that $f(y_0) = 2/(2 - y_1)$. Now let $x_1 = 2 - y_1$. Then $f(x_1f(y_0)) = f(2) = 0$, so $f(x_1 + y_0) = 0$. But $x_1 + y_0 < 2$. Contradiction. So we must have $f(x) = 2/(2 - x)$ for all $x < 2$.

We have thus established that if a function f meets the conditions then it must be defined as above. It remains to prove that with this definition f does meet the conditions. Clearly $f(2) = 0$ and $f(x)$ is non-zero for $0 \leq x < 2$. $f(xf(y)) = f(2x/(2 - y))$. If $2x/(2 - y) \geq 2$, then $f(xf(y)) = 0$. But it also follows that $x + y \geq 2$, and so $f(x + y) = 0$ and hence $f(xf(y)) f(y) = f(x + y)$ as required. If $2x/(2 - y) < 2$, then $f(xf(y)) f(y) = 2/(2 - 2x/(2-y)) 2/(2 - y) = 2/(2 - x - y) = f(x + y)$. So the unique function satisfying the conditions is:

$f(x) = 0$ for $x \geq 2$, and $2/(2 - x)$ for $0 \leq x < 2$.

B3

Answer: yes.

We prove the result by induction on the number n of points. It is clearly true for n = 1. Suppose it is true for all numbers less than n. Pick an arbitrary point P and color it red. Now take a point in the same row and color it white. Take a point in the same column as the new point and color it red. Continue until either you run out of eligible points or you pick a point in the same column as P. The process must terminate because there are only finitely many points. Suppose the last point picked is Q. Let S be the set of points picked.

If Q is in the same column as P, then it is colored white (because the "same row" points are all white, and the "same column" points are all red). Now every row and column contains an equal number of red points of S and of white points of S. By induction we can color the points excluding those in S, then the difference between the numbers of red and white points in each row and column will be unaffected by adding the points in S and so we will have a coloring for the whole set. This completes the induction for the case where Q is in the same column as P.

If it is not, then continue the path backwards from P. In other words, pick a point in the same column as P and color it white. Then pick a point in the same row as the new point and color it red and so on. Continue until either you run out of eligible points or you pick a point to pair with Q. If Q was picked as being in the same row as its predecessor, this means a point in the same column as Q; if Q was picked as being in the same column as its predecessor, this means a point in the same row as Q. Again the process must terminate. Suppose the last point picked is R. Let S be the set of all points picked.

If R pairs with Q, then we can complete the coloring by induction as before. Suppose S does not pair with Q. Then there is a line (meaning a row or column) containing Q and no uncolored points. There is also a line containing R and no uncolored points. These two lines have an excess of one red or one white. All other lines contain equal number of red and white points of S. Now color the points outside S by induction. This gives a coloring for the whole set, because no line with a color excess in S has any points outside S. So we have completed the induction.

IMO 1987

A1

First Solution

We show first that the number of permutations of n objects with no fixed points is $n!(1/0! - 1/1! + 1/2! - \dots + (-1)^n/n!)$. This follows immediately from the law of inclusion and exclusion: let N_i be the number which fix i, N_{ij} the number which fix i and j, and so on. Then N_0, the number with no fixed points, is $n! - \text{all } N_i + \text{all } N_{ij} - \dots + (-1)^n N_{1\dots n}$. But $N_i = (n-1)!$, $N_{ij} = (n-2)!$ and so on. So $N_0 = n!\,(1 - 1/1! + \dots + (-1)^r (n-r)!/(r!\,(n-r)!) + \dots + (-1)^n/n!) = n!\,(1/0! - 1/1! + \dots + (-1)^n/n!)$.

Hence the number of permutations of n objects with exactly r fixed points = no. of ways of choosing the r fixed points x no. of perms of the remaining n − r points with no fixed points = $n!/(r!\,(n-r)!) \times (n-r)!\,(1/0! - 1/1! + \dots + (-1)^{n-r}/(n-r)!)$. Thus we wish to prove that the sum from r = 1 to n of $1/(r-1)!\,(1/0! - 1/1! + \dots + (-1)^{n-r}/(n-r)!)$ is 1. We use induction on n. It is true for n = 1. Suppose it is true for n. Then the sum for n+1 less the sum for n is: $1/0!\,(-1)^n/n! + 1/1!\,(-1)^{n-1}/(n-1)! + \dots + 1/n!\,1/0! = 1/n!\,(1-1)^n = 0$. Hence it is true for n + 1, and hence for all n.

Comment

This is a plodding solution. If you happen to know the result for no fixed points (which many people do), then it is essentially a routine induction.

Second solution

Count all pairs (x, s) where s is a permutation with x a fixed point of x. Clearly, if we fix x, then there are (n−1)! possible permutations s. So the total count is n!. But if we count the number of permutations s with exactly k fixed points, then we get the sum in the question.

Comment

This much more elegant solution is due to Gerhard Wöginger (email 24 Aug 99).

A2

by Gerhard Wöginger

AKL and AML are congruent, so KM is perpendicular to AN and area AKNM = KM.AN/2.
AKLM is cyclic (2 opposite right angles), so angle AKM = angle ALM and hence KM/sin BAC = AM/sin AKM (sine rule) = AM/sin ALM = AL.
ABL and ANC are similar, so AB.AC = AN.AL. Hence area ABC = 1/2 AB.AC sin BAC = 1/2 AN.AL sin BAC = 1/2 AN.KM = area AKNM.

A3

This is an application of the pigeon-hole principle.

Assume first that all x_i are non-negative. Observe that the sum of the x_i is at most $\sqrt{n}$. [This is a well-known variant, $(\sum_{1\le i\le n} x_i)^2 \le n \sum_{1\le i\le n} x_i^2$, of the AM-GM result. See, for example, Arthur Engel, Problem Solving Strategies, Springer 1998, p163, ISBN 0387982191].

Consider the k^n possible values of $\sum_{1\le i\le n} b_i x_i$, where each b_i is an integer in the range [0,k−1]. Each value must lie in the interval [0, k−1 $\sqrt{n}$]. Divide this into k^n-1 equal subintervals. Two values must lie in the same subinterval. Take their difference. Its coefficients are the required a_i. Finally, if any x_i are negative, solve for the absolute values and then flip signs in the a_i.

Comment

This solution is due to Gerhard Woeginger, email 24 Aug 99.

B1

We prove that if f(f(n)) = n + k for all n, where k is a fixed positive integer, then k must be even. If k = 2h, then we may take f(n) = n + h.

Suppose $f(m) = n$ with $m = n \pmod k$. Then by an easy induction on r we find $f(m + kr) = n + kr$, $f(n + kr) = m + k(r+1)$. We show this leads to a contradiction. Suppose $m < n$, so $n = m + ks$ for some $s > 0$. Then $f(n) = f(m + ks) = n + ks$. But $f(n) = m + k$, so $m = n + k(s - 1) \geq n$. Contradiction. So we must have $m \geq n$, so $m = n + ks$ for some $s \geq 0$. But now $f(m + k) = f(n + k(s+1)) = m + k(s + 2)$. But $f(m + k) = n + k$, so $n = m + k(s + 1) > n$. Contradiction.

So if $f(m) = n$, then m and n have different residues mod k. Suppose they have r_1 and r_2 respectively. Then the same induction shows that all sufficiently large $s = r_1 \pmod k$ have $f(s) = r_2 \pmod k$, and that all sufficiently large $s = r_2 \pmod k$ have $f(s) = r_1 \pmod k$. Hence if m has a different residue r mod k, then f(m) cannot have residue r_1 or r_2. For if f(m) had residue r_1, then the same argument would show that all sufficiently large numbers with residue r_1 had $f(m) = r \pmod k$. Thus the residues form pairs, so that if a number is congruent to a particular residue, then f of the number is congruent to the pair of the residue. But this is impossible for k odd.

A better solution by Sawa Pavlov is as follows

Let N be the set of non-negative integers. Put $A = N - f(N)$ (the set of all n such that we cannot find m with $f(m) = n$). Put $B = f(A)$.

Note that f is injective because if $f(n) = f(m)$, then $f(f(n)) = f(f(m))$ so $m = n$. We claim that $B = f(N) - f(f(N))$. Obviously B is a subset of f(N) and if k belongs to B, then it does not belong to $f(f(N))$ since f is injective. Similarly, a member of $f(f(N))$ cannot belong to B.

Clearly A and B are disjoint. They have union $N - f(f(N))$ which is $\{0, 1, 2, \dots, 1986\}$. But since f is injective they have the same number of elements, which is impossible since $\{0, 1, \dots, 1986\}$ has an odd number of elements.

B2

Let x_n be the point with coordinates (n, n^2) for n = 1, 2, 3, ... We show that the distance between any two points is irrational and that the triangle determined by any 3 points has non-zero rational area.

Take $n > m$. $|x_n - x_m|$ is the hypoteneuse of a triangle with sides $n - m$ and $n^2 - m^2 = (n - m)(n + m)$. So $|x_n - x_m| = (n - m)\sqrt{1 + (n+m)^2}$. Now $(n + m)^2 < (n + m)^2 + 1 < (n + m + 1)^2 = (n + m)^2 + 1 + 2(n + m)$, so $(n + m)^2 + 1$ is not a perfect square. Hence its square root is irrational. [For this we may use the classical argument. Let N' be a non-square and suppose $\sqrt{N'}$ is rational. Since N' is a non-square we must be able to find a prime p such that p^{2a+1} divides N' but p^{2a+2} does not divide N' for some $a \geq 0$. Define $N = N'/p^{2a}$. Then $\sqrt{N} = (\sqrt{N'})/p^a$, which is also rational. So we have a prime p such that p divides N, but p^2 does not divide N. Take $\sqrt{N} = r/s$ with r and s relatively prime. So $s^2N = r^2$. Now p must divide r, hence p^2 divides r^2 and so p divides s^2. Hence p divides s.

So r and s have a common factor. Contradiction. Hence non-squares have irrational square roots.]

Now take $a < b < c$. Let B be the point (b, a^2), C the point (c, a^2), and D the point (c, b^2). Area $x_a x_b x_c$ = area $x_a x_c C$ - area $x_a x_b B$ - area $x_b x_c D$ - area $x_b DCB$ = $(c - a)(c^2 - a^2)/2 - (b - a)(b^2 - a^2)/2 - (c - b)(c^2 - b^2)/2 - (c - b)(b^2 - a^2)$ which is rational.

B3

First observe that if m is relatively prime to $b + 1, b + 2, \dots , 2b - 1, 2b$, then it is not divisible by any number less than 2b. For if $c <= b$, then take the largest $j \geq 0$ such that $2^j c \leq b$. Then $2^{j+1}c$ lies in the range $b + 1, \dots , 2b$, so it is relatively prime to m. Hence c is also. If we also have that $(2b + 1)^2 > m$, then we can conclude that m must be prime, since if it were composite it would have a factor $\leq \sqrt{m}$.

Let $n = 3r^2 + h$, where $0 \leq h < 6r + 3$, so that r is the greatest integer less than or equal to $\sqrt{(n/3)}$. We also take $r \geq 1$. That excludes the value $n = 2$, but for $n = 2$, the result is vacuous, so nothing is lost.

Assume that $n + k(k+1)$ is prime for $k = 0, 1, \dots , r$. We show by induction that $N = n + (r + s)(r + s + 1)$ is prime for $s = 1, 2, \dots , n - r - 2$. By the observation above, it is sufficient to show that $(2r + 2s + 1)^2 > N$, and that N is relatively prime to all of $r + s + 1, r + s + 2, \dots , 2r + 2s$. We have $(2r + 2s + 1)^2 = 4r^2 + 8rs + 4s^2 + 4r + 4s + 1$. Since $r, s \geq 1$, we have $4s + 1 > s + 2$, $4s^2 > s^2$, and $6rs > 3r$. Hence $(2r + 2s + 1)^2 > 4r^2 + 2rs + s^2 + 7r + s + 2 = 3r^2 + 6r + 2 + (r + s)(r + s + 1) >= N$.

Now if N has a factor which divides $2r - i$ with i in the range $-2s$ to $r - s - 1$, then so does $N - (i + 2s + 1)(2r - i) = n + (r - i - s - 1)(r - i - s)$ which has the form $n + s'(s'+1)$ with s' in the range 0 to $r + s - 1$. But $n + s'(s' + 1)$ is prime by induction (or absolutely for $s = 1$), so the only way it can have a factor in common with $2r - i$ is if it divides $2r - i$. But $2r - i \leq 2r + 2s \leq 2n - 4 < 2n$ and $n + s'(s' + 1) \geq n$, so if $n + s'(s' + 1)$ has a factor in common with $2r - i$, then it equals $2r - i = s + r + 1 + s'$. Hence $s'^2 = s - (n - r - 1) < 0$, which is not possible. So we can conclude that N is relatively prime to all of $r + s + 1, \dots , 2r + 2s$ and hence prime.

IMO 1988

A1

(i) Let M be the midpoint of BC. Let $PM = x$. Let BC meet the small circle again at Q. Let O be the center of the circles. Since angle APQ = 90 degrees, AQ is a diameter of the small circle, so its length is 2r. Hence $AP^2 = 4r^2 - 4x^2$. $BM^2 = R^2 - OM^2 = R^2 - (r^2 - x^2)$. That is essentially all we need, because we now have: $AB^2 + AC^2 + BC^2 = (AP^2 + (BM - x)^2) + (AP^2 + (BM + x)^2) + 4BM^2 = 2AP^2 + 6BM^2$

$+ 2x^2 = 2(4r^2 - 4x^2) + 6(R^2 - r^2 + x^2) + 2x^2 = 6R^2 + 2r^2$, which is independent of x.

(ii) M is the midpoint of BC and PQ since the circles have a common center. If we shrink the small circle by a factor 2 with P as center, then Q moves to M, and hence the locus of M is the circle diameter OP.

A2

Answer: n even.

Each of the 2n elements of A_i belongs to at least one other A_j because of (iii). But given another A_j it cannot contain more than one element of A_i because of (ii). There are just 2n other A_j available, so each must contain exactly one element of A_i. Hence we can strengthen (iii) to every element of B belongs to exactly two of the As.

This shows that the arrangement is essentially unique. We may call the element of B which belongs to A_i and A_j (i,j). Then A_i contains the 2n elements (i, j) with j not i.

|B| = 1/2 x no. of As x size of each A = n(2n+1). If the labeling with 0s and 1s is possible, then if we list all the elements in each A, n(2n+1) out of the 2n(2n+1) elements have value 0. But each element appears twice in this list, so n(2n+1) must be even. Hence n must be even.

Next part thanks to Stan Dolan

Label (i,j) 0 if j = i−n/2, i−(n/2 − 1), ... , i−1, i+1, i+2, ... , i+n/2 (working mod 2n+1 when necessary). This clearly has the required property.

My original solution was a pedestrian induction:

We show by induction that a labeling is always possible for n even. If n = 2, there is certainly a labeling. For example, we may assign 0 to (1,2), (1,3), (2,4), (3,5), (4,5). Now suppose we have a labeling for n. For n + 2, we label (i , j) 0 if it was labeled 0 for n or if it is:

(i, 2n+2) or (i, 2n+3) for i = 1, 2, ... , n+1
(i, 2n+4) or (i, 2n+5) for i = n+2, n+3, ... , 2n+1
(2n+2, 2n+4), (2n+3, 2n+5), (2n+4,2n+5).

For i = 1, 2, ... n+1, A_i has n elements (i, j) labeled zero with $j \le 2n+1$ and also (i, 2n+2) and (i, 2n+3), giving n+2 in all. For i = n+2, n+3, ... , 2n+1, A_i has n elements (i, j) labeled zero with $j \le 2n+1$ and also (i, 2n+4) and (i, 2n+5), giving n+2 in all. A_{2n+2} has the n+1 elements (i, 2n+2) with i <= n+1 and also (2n+2, 2n+4), giving n+2 in all. A_{2n+3} has the n+1 elements (i, 2n+3) for $i \le n+1$ and also (2n+3, 2n+5), giving n+2 in all. A_{2n+4} has the n elements (i, 2n+4) with $n+2 \le i \le 2n+1$ and also (2n+2, 2n+4) and (2n+4, 2n+5), giving n+2 in all. Finally

A_{2n+5} has the n elements (i, 2n+5) with $n+2 \le i \le 2n+1$ and also (2n+3, 2n+5) and (2n+4, 2n+5), giving n+2 in all.

A3

Answer: 92.

f(n) is always odd. If $n = b_{r+1}b_r \ldots b_2b_1b_0$ in binary and n is odd, so that $b_{r+1} = b_0 = 1$, then $f(n) = b_{r+1}b_1b_2 \ldots b_rb_0$. If n has r+2 binary digits with r > 0, then there are $2^{[(r+1)/2]}$ numbers with the central r digits symmetrical, so that f(n) = n (because we can choose the central digit and those lying before it arbitarily, the rest are then determined). Also there is one number with 1 digit (1) and one number with two digits (3) satisfying f(n) = 1. So we find a total of 1 + 1 + 2 + 2 + 4 + 4 + 8 + 8 + 16 + 16 = 62 numbers in the range 1 to 1023 with f(n) = n. 1988 = 11111000011. So we also have all 32 numbers in the range 1023 to 2047 except for 11111111111 and 11111011111, giving another 30, or 92 in total.

It remains to prove the assertions above. f(n) odd follows by an easy induction. Next we show that if $2^m < 2n+1 < 2^{m+1}$, then $f(2n+1) = f(n) + 2^m$. Again we use induction. It is true for m = 1 (f(3) = f(1) + 2). So suppose it is true for 1, 2, ... , m. Take 4n+1 so that $2^{m+1} < 4n+1 < 2^{m+2}$, then $f(4n+1) = 2f(2n+1) - f(n) = 2(f(n) + 2^m) - f(n) = f(n) + 2^{m+1} = f(2n) + 2^{m+1}$, so it is true for 4n+1. Similarly, if 4n+3 satisfies, $2^{m+1} < 4n+3 < 2^{m+2}$, then $f(4n+3) = 3f(2n+1) - 2f(n) = f(2n+1) + 2(f(n) + 2^m) - 2f(n) = f(2n+1) + 2^{m+1}$, so it is true for 4n+3 and hence for m+1.

Finally, we prove the formula for f(2n+1). Let $2n+1 = b_{r+1}b_r \ldots b_2b_1b_0$ with $b_0 = b_{r+1} = 1$. We use induction on r. So assume it is true for smaller values. Say $b_1 = \ldots = b_s = 0$ and $b_{s+1} = 1$ (we may have s = 0, so that we have simply $b_1 = 1$). Then $n = b_{r+1} \ldots b_1$ and $f(n) = b_{r+1}b_{s+2}b_{s+3} \ldots b_rb_{s+1}$ by induction. So $f(n) + 2^{r+1} = b_{r+1}0 \ldots 0b_{r+1}b_{s+2} \ldots b_rb_{s+1}$, where there are s zeros. But we may write this as $b_{r+1}b_1 \ldots b_sb_{s+1} \ldots b_rb_{r+1}$, since $b_1 = \ldots = b_s = 0$, and $b_{s+1} = b_{r+1} = 1$. But that is the formula for f(2n+1), so we have completed the induction.

B1

Let f(x) = 1/(x − 1) + 2/(x − 2) + 3/(x − 3) + ... + 70/(x − 70). For any integer n, n/(x − n) is strictly monotonically decreasing except at x = n, where it is discontinuous. Hence f(x) is strictly monotonically decreasing except at x = 1, 2, ... , 70. For n = any of 1, 2, ... , 70, n/(x − n) tends to plus infinity as x tends to n from above, whilst the other terms m/(x − m) remain bounded. Hence f(x) tends to plus infinity as x tends to n from above. Similarly, f(x) tends to minus infinity as x tends to n from below. Thus in each of the intervals (n, n+1) for n = 1, ... , 69, f(x) decreases monotonically from plus infinity to minus infinity and hence f(x) = 5/4 has a single foot x_n. Also $f(x) \ge 5/4$ for x in $(n, x_n]$ and $f(x) < 5/4$ for x in $(x_n, n+1)$. If x < 0, then every term is negative and hence f(x) < 0 < 5/4. Finally, as x tends to infinity, every term tends to zero, so f(x) tends to zero. Hence f(x) decreases monotonically from plus infinity to zero over the range [70, infinity]. Hence f(x) = 5/4 has a single root x_{70} in this range and f(x) >= 5/4 for x in (70, x_{70}] and f(x) < 5/4 for x > x_{70}. Thus we have established that f(x) ≥ 5/4 for x in any of the disjoint intervals (1, x_1], (2, x_2], ... , (70, x_{70}] and f(x) < 5/4 elsewhere.

The total length of these intervals is (x_1 − 1) + ... + (x_{70} − 70) = (x_1 + ... + x_{70}) − (1 + ... + 70). The x_i are the roots of the 70th order polynomial obtained from 1/(x − 1) + 2/(x − 2) + 3/(x − 3) + ... + 70/(x − 70) = 5/4 by multiplying both sides by (x − 1) ... (x − 70). The sum of the roots is minus the coefficient of x^{69} divided by the coefficient of x^{70}. The coefficient of x^{70} is simply k, and the coefficient of x^{69} is − (1 + 2 + ... + 70)k − (1 + ... + 70). Hence the sum of the roots is (1 + ... + 70)(1 + k)/k and the total length of the intervals is (1 + ... + 70)/k = 1/2 70·71 4/5 = 28·71 = 1988.

B4

The key is to show that AK = AL = AD. We do this indirectly. Take K' on AB and L' on AC so that AK' = AL' = AD. Let the perpendicular to AB at K' meet the line AD at X. Then the triangles AK'X and ADB are congruent. Let J be the incenter of ADB and let r be the in-radius of ADB. Then J lies on the angle bisector of angle BAD a distance r from the line AD. Hence it is also the incenter of AK'X. Hence JK' bisects the right angle AK'X, so ∠AK'J = 45° and so J lies on K'L'. An exactly similar argument shows that I, the incenter of ADC, also lies on K'L'. Hence we can identify K and K', and L and L'.

The area of AKL is AK·AL/2 = AD^2/2, and the area of ABC is BC·AD/2, so we wish to show that 2AD ≤ BC. Let M be the midpoint of BC. Then AM is the hypoteneuse of AMD, so AM ≥ AD with equality if and only if D = M. Hence 2AD ≤ 2AM = BC with equality if and only if AB = AC.

B3

A little experimentation reveals the following solutions: a, a^3 giving a^2; a^3, a^5 − a giving a^2; and the recursive a_1 = 2, b_1 = 8, a_{n+1} = b_n, b_{n+1} = 4b_n − a_n giving 4. The latter may lead us to: if $a^2 + b^2$ = k(ab + 1), then take A = b, B = kb − a, and then $A^2 + B^2$ = k(AB + 1). Finally, we may notice that this can be used to go down as well as up.

So starting again suppose that a, b, k is a solution in positive integers to $a^2 + b^2$ = k(ab + 1). If a = b, then $2a^2$ = k(a^2 + 1). So a^2 must divide k. But that implies that a = b = k = 1. Let us assume we do not have this trivial solution, so we may take a < b. We also show that a^3 > b. For (b/a − 1/a)(ab + 1) = b^2 + b/a − b − 1/a < b^2 < $a^2 + b^2$. So k > b/a − 1/a. But if a^3 < b, then b/a (ab + 1) > $b^2 + a^2$, so k < b/a. But now b > ak and < ak + 1, which is impossible. It follows that k ≥ b/a.

Now define $A = ka - b$, $B = a$. Then we can easily verify that A, B, k also satisfies $a^2 + b^2 = k(ab + 1)$, and B and k are positive integers. Also $a < b$ implies $a^2 + b^2 < ab + b^2 < ab + b^2 + 1 + b/a = (ab + 1)(1 + b/a)$, and hence $k < 1 + b/a$, so $ka - b < a$. Finally, since $k > b/a$, $ka - b \geq 0$. If $ka - b > 0$, then we have another smaller solution, in which case we can repeat the process. But we cannot have an infinite sequence of decreasing numbers all greater than zero, so we must eventually get $A = ka - b = 0$. But now $A^2 + B^2 = k(AB + 1)$, so $k = B^2$. k was unchanged during the descent, so k is a perfect square.

A slightly neater variation on this is due to Stan Dolan

As above take $a^2 + b^2 = k(ab + 1)$, so a, b, and k are all positive integers. Now fixing k take positive integers A, B such that $A^2 + B^2 = k(AB + 1)$ (*) and min(A,B) is as small as possible. Assume $B \leq A$. Regarding (*) as a quadratic for A, we see that the other root C satisfies $A + C = kB$, $AC = B^2 - k$. The second equation implies that $C = B^2/A - k/A < B$. So C cannot be a positive integer (or the solution C, B would have $\min(C,B) < \min(A,B)$). But we have $(A+1)(C+1) = A+C + AC + 1 = B^2 + (B-1)k + 1 > 0$, so $C > -1$. $C = kB - A$ is an integer, so $C = 0$. Hence $k = B^2$.

Note that jumping straight to the minimal without the infinite descent avoids some of the verification needed in the infinite descent.

IMO 1992

A1

Answer: $a = 2$, $b = 4$, $c = 8$; or $a = 3$, $b = 5$, $c = 15$.

Let $k = 2^{1/3}$. If $a \geq 5$, then $k(a - 1) > a$. [Check: $(k(a - 1)^3 - a^3 = a^3 - 6a^2 + 6a - 2$. For $a \geq 6$, $a^3 \geq 6a^2$ and $6a > 2$, so we only need to check $a = 5$: $125 - 150 + 30 - 2 = 3$.] We know that $c > b > a$, so if $a \geq 5$, then $2(a - 1)(b - 1)(c - 1) > abc > abc - 1$. So we must have $a = 2$, 3 or 4.

Suppose $abc - 1 = n(a - 1)(b - 1)(c - 1)$. We consider separately the cases $n = 1$, $n = 2$ and $n \geq 3$. If $n = 1$, then $a + b + c = ab + bc + ca$. But that is impossible, because a, b, c are all greater than 1 and so $a < ab$, $b < bc$ and $c < ca$.

Suppose $n = 2$. Then $abc - 1$ is even, so all a, b, c are odd. In particular, $a = 3$. So we have $4(b - 1)(c - 1) = 3bc - 1$, and hence $bc + 5 = 4b + 4c$. If $b >= 9$, then $bc >= 9c > 4c + 4b$. So we must have $b = 5$ or 7. If $b = 5$, then we find $c = 15$, which gives a solution. If $b = 7$, then we find $c = 23/3$ which is not a solution.

The remaining case is n >= 3. If a = 2, we have n(bc − b − c + 1) = 2bc − 1, or (n − 2)bc + (n + 1) = nb + nc. But b ≥ 3, so (n − 2)bc ≥ (3n − 6)c ≥ 2nc for n ≥ 6, so we must have n = 3, 4 or 5. If n = 3, then bc + 4 = 3b + 3c. If b >= 6, then bc ≥ 6c > 3b + 3c, so b = 3, 4 or 5. Checking we find only b = 4 gives a solution: a = 2, b = 4, c = 8. If n = 4, then (n − 2)bc, nb and nc are all even, but (n + 1) is odd, so there is no solution. If n = 5, then 3bc + 6 = 5b + 5c. b = 3 gives c = 9/4, which is not a solution. b >= 4 gives 3bc > 10c > 5b + 5c, so there are no solutions.

If a = 3, we have 2n(bc − b − c + 1) = 3bc − 1, or (2n − 3)bc + (2n + 1) = 2nb + 2nc. But b ≥ 4, so (2n − 3)bc ≥ (8n − 12)c ≥ 4nc > 2nc + 2nb. So there are no solutions. Similarly, if a = 4, we have (3n − 4)bc + (3n + 1) = 3nb + 3nc. But b ≥ 4, so (3n − 4)bc ≥ (12n − 16)c > 6nc > 3nb + 3nc, so there are no solutions.

A2

The first step is to establish that f(0) = 0. Putting x = y = 0, and f(0) = t, we get $f(t) = t^2$. Also, $f(x^2+t) = f(x)^2$, and $f(f(x)) = x + t^2$. We now evaluate $f(t^2+f(1)^2)$ two ways. First, it is $f(f(1)^2 + f(t)) = t + f(f(1))^2 = t + (1 + t^2)^2 = 1 + t + 2t^2 + t^4$. Second, it is $f(t^2 + f(1 + t)) = 1 + t + f(t)^2 = 1 + t + t^4$. So t = 0, as required.

It follows immediately that f(f(x)) = x, and $f(x^2) = f(x)^2$. Given any y, let z = f(y). Then y = f(z), so $f(x^2 + y) = z + f(x)^2 = f(y) + f(x)^2$. Now given any positive x, take z so that $x = z^2$. Then $f(x + y) = f(z^2 + y) = f(y) + f(z)^2 = f(y) + f(z^2) = f(x) + f(y)$. Putting y = −x, we get 0 = f(0) = f(x + −x) = f(x) + f(−x). Hence f(−x) = −f(x). It follows that f(x + y) = f(x) + f(y) and f(x − y) = f(x) − f(y) hold for all x, y.

Take any x. Let f(x) = y. If y > x, then let z = y − x. f(z) = f(y − x) = f(y) − f(x) = x − y = −z. If y < x, then let z = x − y and f(z) = f(x − y) = f(x) − f(y) = y − x. In either case we get some z > 0 with f(z) = −z < 0. But now take w so that $w^2 = z$, then $f(z) = f(w^2) = f(w)^2 >= 0$. Contradiction. So we must have f(x) = x.

A3

Solution

by Gerhard Wöginger

We show that for n = 32 we can find a coloring without a monochrome triangle. Take two squares $R_1R_2R_3R_4$ and $B_1B_2B_3B_4$. Leave the diagonals of each square uncolored, color the remaining edges of R red and the remaining edges of B blue. Color blue all the edges from the ninth point X to the red square, and red all the edges from X to the blue square. Color R_iB_j red if i and j have the same parity and blue otherwise.

Clearly X is not the vertex of a monochrome square, because if XY and XZ are the same color then, YZ is either uncolored or the opposite color. There is no triangle within the red square or the blue square, and hence no monochrome triangle. It remains to consider triangles of the form $R_iR_jB_k$ and $B_iB_jR_k$. But if i and j have the same parity, then R_iR_j is uncolored (and similarly B_iB_j), whereas if they have opposite parity, then R_iB_k and R_jB_k have opposite colors (and similarly B_iR_k and B_jR_k).

It remains to show that for n = 33 we can always find a monochrome triangle. There are three uncolored edges. Take a point on each of the uncolored edges. The edges between the remaining 6 points must all be colored. Take one of these, X. At least 3 of the 5 edges to X, say XA, XB, XC must be the same color (say red). If AB is also red, then XAB is monochrome. Similarly, for BC and CA. But if AB, BC and CA are all blue, then ABC is monochrome.

B1

Answer: Let X be the point where C meets L, let O be the center of C, let XO cut C gain at Z, and take Y on QR so that M be the midpoint of XY. Let L' be the line YZ. The locus is the open ray from Z along L' on the opposite side to Y.

mainly by Gerhard Wöginger, Technical University, Graz (I filled in a few details)

Let C' be the circle on the other side of QR to C which also touches the segment QR and the lines PQ and QR. Let C' touch QR at Y'. If we take an expansion (technically, homothecy) center P, factor PY'/PZ, then C goes to C', the tangent to C at Z goes to the line QR, and hence Z goes to Y'. But it is easy to show that QX = RY'.

We focus on the QORO'. Evidently X,Y' are the feet of the perpendiculars from O, O' respectively to QR. Also, OQO' = ORO' = 90. So QY'O' and OXQ are similar, and hence QY'/Y'O' = OX/XQ. Also RXO and O'Y'R are similar, so RX/XO = O'Y'/Y'R. Hence QY'·XQ = OX·O'Y' = RX·Y'R. Hence QX/RX = QX/(QR – QX) = RY'/(QR – RY') = RY'/QY'. Hence QX = RY'.

But QX = RY by construction (M is the midpoint of XY and QR), so Y = Y'. Hence P lies on the open ray as claimed. Conversely, if we take P on this ray, then by the same argument QX = RY. But M is the midpoint of XY, so M must also be the midpoint of QR, so the locus is the entire (open) ray.

Gerhard only found this after Theo Koupelis, University of Wisconsin, Marathon had already supplied the following analytic solution.

Take Cartesian coordinates with origin X, so that M is (a, 0) and O is (0, R). Let R be the point (b, 0) (we take a, b >= 0). Then Q is the point (2a – b, 0), and Y is (2a, 0). Let angle XRO be θ. Then $\tan \theta = R/b$ and angle PRX = 2θ, so $\tan PRX = 2 \tan \theta/(1 - \tan^2\theta) = 2Rb/(b^2 - R^2)$. Similarly, $\tan PQX = 2R(b - 2a)/((b - 2a)^2 - R^2)$.

If P has coordinates (A, B), then B/(b − A) = tan PRX, and B/(b − 2a + x) = tan PQX. So we have two simultaneous equations for A and B. Solving, and simplifying slightly, we find $A = -2aR^2/(b^2 - 2ab - R^2)$, $B = 2b(b - 2a)R/(b^2 - 2ab - R^2)$. (*)

We may now check that B/(2a − A) = R/a, so P lies on YZ as claimed. So we have shown that the locus is a subset of the line YZ. But since $b^2 - 2ab - R^2$ maps the open interval $(a + \sqrt{(a^2 + R^2)}, \infty)$ onto the open interval $(0, \infty)$, (*) shows that we can obtain any value A in the open interval $(-\infty,0)$ by a suitable choice of b, and hence any point P on the ray (except its endpoint Z) by a suitable choice of R.

B2

by Gerhard Wöginger

Induction on the number of different z-coordinates in S.

For 1, it is sufficient to note that $S = S_z$ and $|S| \le |S_x|\ |S_y|$ (at most $|S_x|$ points of S project onto each of the points of S_y).

In the general case, take a horizontal (constant z) plane dividing S into two non-empty parts T and U. Clearly, $|S| = |T| + |U|$, $|S_x| = |T_x| + |U_x|$, and $|S_y| = |T_y| + |U_y|$.
By induction, $|S| = |T| + |U| \le (|T_x|\ |T_y|\ |T_z|)^{1/2} + (|U_x|\ |U_y|\ |U_z|)^{1/2}$. But $|T_z|, |U_z| \le |S_z|$, and for any positive a, b, c, d we have $(a\ b)^{1/2} + (c\ d)^{1/2} \le ((a + c)\ (b + d))^{1/2}$ (square!).
Hence $|S| \le |S_z|^{1/2}((|T_x| + |U_x|)\ (|T_y| + |U_y|))^{1/2} = (|S_x|\ |S_y|\ |S_z|)^{1/2}$.

B3

(a) Let $N = n^2$. Suppose we could express N as a sum of N − 13 squares. Let the number of 4s be a, the number of 9s be b and so on. Then we have 13 = 3a + 8b + 15c + … . Hence c, d, … must all be zero. But neither 13 nor 8 is a multiple of 3, so there are no solutions. Hence $S(n) \le N - 14$.

A little experimentation shows that the problem is getting started. Most squares cannot be expressed as a sum of two squares. For $N = 13^2 = 169$, we find: 169 = 9 + 4 + 4 + 152 1s, a sum of 155 = N − 14 squares. By grouping four 1s into a 4 repeatedly, we obtain all multiples of 3 plus 2 down to 41 (169 = 9 + 40 4s). Then grouping four 4s into a 16 gives us 38, 35, … , 11 (169 = 10 16s + 9). Grouping four 16s into a 64 gives us 8 and 5. We obtain the last number congruent to 2 mod 3 by the decomposition: $169 = 12^2 + 5^2$.

For the numbers congruent to 1 mod 3, we start with N − 15 = 154 squares: 169 = 5 4s + 149 1s. Grouping as before gives us all 3m + 1 down to 7: 169 = 64 + 64 + 16 + 16 + 4 + 4 + 1. We may use $169 = 10^2 + 8^2 + 2^2 + 1^2$ for 4.

For multiples of 3, we start with N − 16 = 153 squares: 169 = 9 + 9 + 151 1s. Grouping as before gives us all multiples of 3 down to 9: 169 = 64 + 64 + 16 + 9 + 9 + 4 + 1 + 1 + 1. Finally, we may take $169 = 12^2 + 4^2 + 3^2$ for 3 and split the 4^2 to get $169 = 12^2 + 3^2 + 2^2 + 2^2 + 2^2 + 2^2$ for 6. That completes the demonstration that we can write 13^2 as a sum of k positive squares for all k <= $S(13) = 13^2 - 14$.

We now show how to use the expressions for 13^2 to derive further N. For any N, the grouping technique gives us the high k. Simply grouping 1s into 4s takes us down: from 9 + 4 + 4 + (N−17) 1s to (N−14)/4 + 6 < N/2 or below; from 4 + 4 + 4 + 4 + 4 + (N−20) 1s to (N−23)/4 + 8 < N/2 or below; from 9 + 9 + (N−18) 1s to (N−21)/4 + 5 < N/2 or below. So we can certainly get all k in the range (N/2 to N−14) by this approach. Now suppose that we already have a complete set of expressions for N_1 and for N_2 (where we may have $N_1 = N_2$). Consider $N_3 = N_1N_2$. Writing $N_3 = N_1$(an expression for N_2 as a sum of k squares) gives N_3 as a sum of 1 thru k_2 squares, where $k_2 = N_2 - 14$ squares (since N_1 is a square). Now express N_1 as a sum of two squares: $n_1^2 + n_2^2$. We have $N_3 = n_1^2$(a sum of k_2 squares) + n_2^2(a sum of k squares). This gives N_3 as a sum of $k_2 + 1$ thru $2k_2$ squares. Continuing in this way gives N_3 as a sum of 1 thru k_1k_2 squares. But $k_i = N_i - 14 > 2/3\ N_i$, so $k_1k_2 > N_3/2$. So when combined with the top down grouping we get a complete set of expressions for N_3.

This shows that there are infinitely many squares N with a complete set of expressions, for example we may take N = the squares of 13, 13^2, 13^3, ...

IMO 1993

A1

Suppose $f(x) = (x^r + a_{r-1}x^{r-1} + \dots + a_1x \pm 3)(x^s + b_{s-1}x^{s-1} + \dots + b_1x \pm 1)$. We show that all the a's are divisible by 3 and use that to establish a contradiction.

First, r and s must be greater than 1. For if r = 1, then ± 3 is a root, so if n is even, we would have $0 = 3^n \pm 5 \cdot 3^{n-1} + 3 = 3^{n-1}(3 \pm 5) + 3$, which is false since 3 ± 5 = 8 or −2. Similarly if n is odd we would have $0 = 3^{n-1}(\pm 3 + 5) + 3$, which is false since ±3 + 5 = 8 or 2. If s = 1, then ±1 is a root and we obtain a contradiction in the same way.

So $r \le n - 2$, and hence the coefficients of x, x^2, ... , x^r are all zero. Since the coefficient of x is zero, we have: $a_1 \pm 3b_1 = 0$, so a_1 is divisible by 3. We can now proceed by induction. Assume $a_1, \dots, a_t$ are all divisible by 3. Then consider the coefficient of x^{t+1}. If $s-1 \ge t+1$, then a_{t+1} = linear combination of $a_1, \dots, a_t \pm 3b_{t+1}$. If $s-1 < t+1$, then a_{t+1} = linear combination of some or all of $a_1, \dots, a_t$. Either way, a_{t+1} is divisible by 3. So considering the coefficients of x, x^2, ... , x^{r-1} gives us that all the a's are multiples of 3. Now consider the coefficient of x^r, which is also zero. It is a sum of terms which are multiples of 3 plus ±1, so it is not zero. Contradiction. Hence the factorization is not possible.

A2

By Glen Ong, Oracle Corporation

Take B' so that CB = CB', □BCB' = 90° and B' is on the opposite side of BC to A. It is easy to check that ADB, ACB' are similar and DAC, BAB' are similar. Hence AB/BD = AB'/B'C and CD/AC = BB'/AB'. It follows that the ratio given is BB'/B'C which is √2.

Take XD the tangent to the circumcircle of ADC at D, so that XD is in the □ADB. Similarly, take YD the tangent to the circumcircle BDC at D. Then □ADX = □ACD, □BDY = □BCD, so □ADX + □BDY = □ACB and hence □XDY = □ADB − (□ADX + □BDY) = □ADB − □ACB = 90°. In other words the tangents to the circumcircles at D are perpendicular. Hence, by symmetry (reflecting in the line of centers) the tangents at C are perpendicular.

Theo Koupelis, University of Wisconsin, Marathon provided a similar solution (about 10 minutes later!) taking the point B' so that □BDB' = 90°, BD = B'D and □B'DA = □ACB. DAC, B'AB are similar; and ABC, AB'D are similar.

Marcin Mazur, University of Illinois at Urbana-Champaign provided the first solution I received (about 10 minutes earlier!) using the generalized Ptolemy's equality (as opposed to the easier equality), but I do not know of a slick proof of this, so I prefer the proof above.

A3

We show first that the game can end with only one piece if n is not a multiple of 3. Note first that the result is true for n = 2 or 4.

n=2

```
X X    . . X    . . X    . . .

X X    X X      . . X    . . .

                               X
```

n = 4

```
              X         X         X         X         X

X X X X   X X . X   X X . X   . . X X   . . X X   . X . .
```

```
X X X X   X X . X   . . X X   . . X X   X . X X   X . X X

X X X X   X X X X   X X X X   X X X X   . X X X   . X X X

X X X X   X X X X   X X X X   X X X X   . X X X   . X X X
```

```
    X         X         X

. X . .   . X . .   . X X .   . X . .   . X . .   . . . .

X X . .   . . X .   . . . .   . . X .   . X X .   . . X .

. X X X   . X X X   . X . X   . X . X   . . . X   . X . X

. X X X   . X X X   . X X X   . X X X   . . X X   . . X X
```

```
. . . .   . . . .   . . . .   . . . .

. . X X   . X . .   . . . .   . . . .

. X . .   . X . .   . . . .   . . . .

. . X .   . . X .   . X X .   . . . X
```

The key technique is the following three moves which can be used to wipe out three adjacent pieces on the border provided there are pieces behind them:

```
X X X     X X .     X X .     X X X

X X X     X X .     . . X     . . .

              X         X
```

We can use this technique to reduce (r + 3) x s rectangle to an r x s rectangle. There is a slight wrinkle for the last two rows of three:

```
X X X X     X X . X     . . X X     . . X X     . . . X     . . . X

X X X X     X X . X     . . X X     . . X X     . . . X     . . . X

. . . X     . . X X     . . X X     . X . .     . X X .     . . . X
```

Thus we can reduce a square side 3n+2 to a 2 x (3n+2) rectangle. We now show how to wipe out the rectangle. First, we change the 2 x 2 rectangle at one end into a single piece alongside the (now) 2 x 3n rectangle:

```
 X X    . .     . .

 X X    . .     . .

        X X      X
```

Then we use the following technique to shorten the rectangle by 3:

```
X X X    X . X    X . X    . . .    . . .

X X X    X . X    X . X    . . .    . . .

X        X X    X .      X X . X       . X
```

That completes the case of the square side 3n+2. For the square side 3n+1 we can use the technique for removing 3 x r rectangles to reduce it to a 4 x 4 square and then use the technique above for the 4 x 4 rectangle.

Finally, we use a parity argument to show that if n is a multiple of 3, then the square side n cannot be reduced to a single piece. Color the board with 3 colors, red, white and blue:

```
R W B R W B R W B ...

W B R W B R W B R ...

B R W B R W B R W ...

R W B R W B R W B ...

 ...
```

Let suppose that the single piece is on a red square. Let A be the number of moves onto a red square, B the number of moves onto a white square and C the number of moves onto a blue square. A move onto a red square increases the number of pieces on red squares by 1, reduces the number of pieces on white squares by 1, and reduces the number of pieces on blue squares by 1. Let n = 3m. Then there are initially m pieces on red squares, m on white and m on blue. Thus we have:

$- A + B + C = m-1$; $A - B + C = m$; $A + B - C = m$.

Solving, we get $A = m$, $B = m - 1/2$, $C = m - 1/2$. But the number of moves of each type must be integral, so it is not possible to reduce the number of pieces to one if n is a multiple of 3.

B1

The length of an altitude is twice the area divided by the length of the corresponding side. Suppose that BC is the longest side of the triangle ABC. Then m(ABC) = area ABC/BC. [If A = B = C, so that BC = 0, then the result is trivially true.]

Consider first the case of X inside ABC. Then area ABC = area ABX + area AXC + area XBC, so m(ABC)/2 = area ABX/BC + area AXC/BC + area XBC/BC. We now claim that the longest side of ABX is at most BC, and similarly for AXC and XBC. It then follows at once that area ABX/BC $\le$ area ABX/longest side of ABX = m(ABX)/2 and the result follows (for points X inside ABC).

The claim follows from the following lemma. If Y lies between D and E, then FY is less than the greater than FD and FE. Proof: let H be the foot of the perpendicular from F to DE. One of D and E must lie on the opposite side of Y to H. Suppose it is D. Then FD = FH/cos HFD > FH/cos HFY = FY. Returning to ABCX, let CX meet AB at Y. Consider the three sides of ABX. By definition AB $\le$ BC. By the lemma AX is smaller than the larger of AC and AY, both of which do not exceed BC. Hence AX $\le$ BC. Similarly BX $\le$ BC.

It remains to consider X outside ABC. Let AX meet AC at O. We show that the sum of the smallest altitudes of ABY and BCY is at least the sum of the smallest altitudes of ABO and ACO. The result then follows, since we already have the result for X = O. The altitude from A in ABX is the same as the altitude from A in ABO. The altitude from X in ABX is clearly longer than the altitude from O in ABO (let the altitudes meet the line AB at Q and R respectively, then triangles BOR and BXQ are similar, so XQ = OR·BX/BO > OR). Finally, let k be the line through A parallel to BX, then the altitude from B in ABX either crosses k before it meets AX, or crosses AC before it crosses AX. If the former, then it is longer than the perpendicular from B to k, which equals the altitude from A to BO. If the latter, then it is longer than the altitude from B to AO. Thus each of the altitudes in ABX is longer than an altitude in ABO, so m(ABX) > m(ABO).

B2

Yes: $f(n) = [g*n + ½]$, where $g = (1 + \sqrt{5})/2 = 1.618 \ldots$.

This simple and elegant solution is due to Suengchur Pyun

Let $g(n) = [g*n + ½]$. Obviously $g(1) = 2$. Also $g(n+1) = g(n) + 1$ or $g(n) + 2$, so certainly $g(n+1) > g(n)$.

Consider $d(n) = g* [g*n + ½] + ½ - ([g*n + ½] + n)$. We show that it is between 0 and 1. It follows immediately that $g(g(n)) = g(n) + n$, as required.

Certainly, $[g*n + ½] > g*n - ½$, so $d(n) > 1 - g/2 > 0$ (the n term has coefficient $g^2 - g - 1$ which is zero). Similarly, $[g*n + ½] \le g*n + ½$, so $d(n) \le g/2 < 1$, which completes the proof.

I originally put up the much clumsier result following:

Take $n = b_r b_{r-1} \dots b_0$ in the Fibonacci base. Then $f(n) = b_r b_{r-1} \dots b_0 0$. This satisfies the required conditions.

Let $u_0 = 1$, $u_1 = 2$, ... , $u_n=u_{n-1}+u_{n-2}$, ... be the Fibonacci numbers. We say $n = b_r b_{r-1} \dots b_0$ in the Fibonacci base if $b_r = 1$, every other $b_i = 0$ or 1, no two adjacent b_i are non-zero, and $n = b_r u_r + \dots + b_0 u_0$. For example, 28 = 1001010 because 28 = 21 + 5 + 2.

We have to show that every n has a unique expression of this type. We show first by induction that it has at least one expression of this type. Clearly that is true for n = 1. Take u_r to be the largest Fibonacci number $\le n$. Then by induction we have an expression for $n - u_r$. The leading term cannot be u_i for $i > r - 2$, for then we would have $n >= u_r + u_{r-1} = u_{r+1}$. So adding u_r to the expression for $n - u_r$ gives us an expression of the required type for n, which completes the induction.

We show that $u_r + u_{r-2} + u_{r-4} + \dots = u_{r+1} - 1$. Again we use induction. It is true for r = 1 and 2. Suppose it is true for r – 1, then $u_{r+1} + u_{r-1} + \dots = u_{r+2} - u_r + u_{r-1} + u_{r-3} + \dots = u_{r+2} - u_r + u_r - 1 = u_{r+2} - 1$. So it is true for r + 1. Hence it is true for all r. Now we can prove that the expression for n is unique. It is for n = 1. So assume it is for all numbers < n, but that the expression for n is not unique, so that we have $n = u_r$ + more terms = u_s + more terms. If r = s, then the expression for $n - u_r$ is not unique. Contradiction. So suppose r > s. But now the second expression is at most $u_{s+1} - 1$ which is less than u_r. So the expression for n must be unique and the induction is complete.

It remains to show that f satisfies the required conditions. Evidently if $n = 1 = u_0$, then $f(n) = u_1 = 2$, as required. If $n = u_{a1} + \dots + u_{ar}$, then $f(n) = u_{a1+1} + \dots + u_{ar+1}$ and $f(f(n)) = u_{a1+2} + \dots + u_{ar+2}$. So $f(n) + n = (u_{a1} + u_{a1+1}) + \dots + (u_{ar} + u_{ar+1}) = f(f(n))$.

B3

(a) The process cannot terminate, because before the last move a single lamp would have been on. But the last move could not have turned it off, because the adjacent lamp was off. There are only finitely many states (each lamp is on or off and the next move can be at one of finitely many lamps), hence the process must repeat. The outcome of each step is uniquely determined by the state, so either the process moves around a single large loop, or there is an initial sequence of steps as far as state k and then the process goes around a loop back to k. However, the latter is not possible because then state k would have had two different precursors. But a state has only one possible precursor which can be found by toggling the lamp at the current position if the previous lamp is on and then moving the position back one. Hence the process must move around a single large loop, and hence it must return to the initial state.

(b) Represent a lamp by X when on, by – when not. For 4 lamps the starting situation and the situation after 4, 8, 12, 16 steps is as follows:

X X X X

– X – X

X – – X

– – – X

X X X –

On its first move lamp n–2 is switched off and then remains off until each lamp has had n–1 moves. Hence for each of its first n–1 moves lamp n–1 is not toggled and it retains its initial state. After each lamp has had n–1 moves, all of lamps 1 to n–2 are off. Finally over the next n–1 moves, lamps 1 to n–2 are turned on, so that all the lamps are on. We show by induction on k that these statements are all true for $n = 2^k$. By inspection, they are true for k = 2. So suppose they are true for k and consider $2n = 2^{k+1}$ lamps. For the first n–1 moves of each lamp the n left-hand and the n right-hand lamps are effectively insulated. Lamps n–1 and 2n–1 remain on. Lamp 2n–1 being on means that lamps 0 to n–2 are in just the same situation that they would be with a set of only n lamps. Similarly, lamp n–1 being on means that lamps n to 2n–2 are in the same situation that they would be with a set of only n lamps. Hence after each lamp has had n–1 moves, all the lamps are off except for n–1 and 2n–1. In the next n moves lamps 1 to n–2 are turned on, lamp n–1 is turned off, lamps n to 2n–2 remain off, and lamp 2n–1 remains on. For the next n–1 moves for each lamp, lamp n–1 is not toggled, so it remains off. Hence all of n to 2n–2 also remain off and 2n–1 remains on. Lamps 0 to n–2 go through the same sequence as for a set of n lamps. Hence after these n–1 moves for each lamp, all the lamps are off, except for 2n–1. Finally, over the next 2n–1 moves, lamps 0 to 2n–2 are turned on. This completes the induction. Counting moves, we see that there are n–1 sets of n moves, followed by n–1 moves, a total of $n^2 - 1$.

(c) We show by induction on the number of moves that for $n = 2^k + 1$ lamps after each lamp has had m moves, for $i = 0, 1, \dots, 2^k - 2$, lamp i+2 is in the same state as lamp i is after each lamp has had m moves in a set of $n - 1 = 2^k$ lamps (we refer to this as lamp i in the *reduced* case). Lamp 0 is off and lamp 1 is on. It is easy to see that this is true for m = 1 (in both cases odd numbered lamps are on and even numbered lamps are off). Suppose it is true for m. Lamp 2 has the same state as lamp 0 in the reduced case and both toggle since their predecessor lamps are on. Hence lamps 3 to n − 1 behave the same as lamps 1 to n − 3 in the reduced case. That means that lamp n − 1 remains off. Hence lamp 0 does not toggle on its m+1th move and remains off. Hence lamp 1 does not toggle on its m+1th move and remains on. The induction stops working when lamp n − 2 toggles on its nth move in the reduced case, but it works up to and including m = n − 2. So after n − 2 moves for each lamp all lamps are off except lamp 1. In the next two moves nothing happens, then in the following n − 1 moves lamps 2 to n − 1 and lamp 0 are turned on. So all the lamps are on after a total of $(n - 2)n + n + 1 = n^2 + n + 1$ moves.

IMO 1994

A1

Take $a_1 < a_2 < \dots < a_m$. Take $k \le (m+1)/2$. We show that $a_k + a_{m-k+1} \ge n + 1$. If not, then the k distinct numbers $a_1 + a_{m-k+1}, a_2 + a_{m-k+1}, \dots, a_k + a_{m-k+1}$ are all $\le n$ and hence equal to some a_i. But they are all greater than a_{m-k+1}, so each i satisfies $m-k+2 \le i \le m$, which is impossible since there are only k−1 available numbers in the range.

A2

Assume OQ is perpendicular to EF. Then $\angle EBO = \angle EQO = 90^\circ$, so EBOQ is cyclic. Hence $\angle OEQ = \angle OBQ$. Also $\angle OQF = \angle OCF = 90^\circ$, so OQCF is cyclic. Hence $\angle OFQ = \angle OCQ$. But $\angle OCQ = \angle OBQ$ since ABC is isosceles. Hence $\angle OEQ = \angle OFQ$, so OE = OF, so triangles OEQ and OFQ are congruent and QE = QF.

Assume QE = QF. If OQ is not perpendicular to EF, then take E'F' through Q perpendicular to OQ with E' on AB and F' on AC. Then QE' = QF', so triangles QEE' and QFF' are congruent. Hence $\angle QEE' = \angle QFF'$. So CA and AB make the same angles with EF and hence are parallel. Contradiction. So OQ is perpendicular to EF.

A3

$2, 4, \dots, n(n-1)/2 + 1, \dots$.

To get a feel, we calculate the first few values of f explicitly:

f(2) = 0, f(3) = 0
f(4) = f(5) = 1, [7 = 111]
f(6) = 2, [7 = 111, 11 = 1011]
f(7) = f(8) = f(9) = 3 [11 = 1011, 13 = 1101, 14 = 1110]
f(10) = 4 [11, 13, 14, 19 = 10011]
f(11) = f(12) = 5 [13, 14, 19, 21 = 10101, 22 = 10110]
f(13) = 6 [14, 19, 21, 22, 25 = 11001, 26 = 11010]

We show that f(k+1) = f(k) or f(k) + 1. The set for k+1 has the additional elements 2k+1 and 2k+2 and it loses the element k+1. But the binary expression for 2k+2 is the same as that for k+1 with the addition of a zero at the end, so 2k+2 and k+1 have the same number of 1s. So if 2k+1 has three 1s, then f(k+1) = f(k) + 1, otherwise f(k+1) = f(k). Now clearly an infinite number of numbers 2k+1 have three 1s, (all numbers $2^r + 2^s + 1$ for r > s > 0). So f(k) increases without limit, and since it only moves up in increments of 1, it never skips a number. In other words, given any positive integer m we can find k so that f(k) = m.

From the analysis in the last paragraph we can only have a single k with f(k) = m if both 2k–1 and 2k+1 have three 1s, or in other words if both k–1 and k have two 1s. Evidently this happens when k–1 has the form $2^n + 1$. This determines the k, namely $2^n + 2$, but we need to determine the corresponding m = f(k). It is the number of elements of $\{2^n+3, 2^n+4, \dots, 2^{n+1}+4\}$ which have three 1s. Elements with three 1s are either $2^n+2^r+2^s$ with $0 \le r < s < n$, or $2^{n+1}+3$. So there are m= n(n–1)/2 + 1 of them. As a check, for n = 2, we have k = 2^2+2 = 6, m = 2, and for n = 3, we have k = 2^3+2 = 10, m = 4, which agrees with the f(6) = 2, f(10) = 4 found earlier.

B1

Answer

(1, 2), (1, 3), (2, 1), (2, 2), (2, 5), (3, 1), (3, 5), (5, 2), (5, 3).

We start by checking small values of n. n = 1 gives $n^3 + 1 = 2$, so m = 2 or 3, giving the solutions (2, 1) and (3, 1). Similarly, n = 2 gives $n^3 + 1 = 9$, so 2m–1 = 1, 3 or 9, giving the solutions (1, 2), (2, 2), (5, 2). Similarly, n = 3 gives $n^3 + 1$ = 28, so 3m – 1 = 2, 14, giving the solutions (1, 3), (5, 3). So we assume hereafter that n > 3.

Let $n^3 + 1 = (mn - 1)h$. Then we must have h = –1 (mod n). Put h = kn – 1. Then $n^3 + 1 = mkn^2 - (m + k)n + 1$. Hence $n^2 = mkn - (m + k)$. (*) Hence n divides m + k. If $m + k \ge 3n$, then since n > 3 we have at least one of m, $k \ge n + 2$. But then $(mn - 1)(kn - 1) \ge (n^2 + 2n - 1)(n - 1) = n^3 + n^2 - 3n + 1 = (n^3 + 1) + n(n - 3) > n^3 + 1$. So we must have m + k = n or 2n.

Consider first m + k = n. We may take m ≥ k (provided that we remember that if m is a solution, then so is n – m). So (*) gives n = m(n – m) – 1. Clearly m = n – 1 is not a solution. If m = n – 2, then n = 2(n – 2) – 1, so n = 5. This gives the two solutions (m, n) = (2, 5) and (3, 5). If m < n – 2 then n – m ≥ 3 and so m(n – m) – 1 ≥ 3m – 1 ≥ 3n/2 – 1 > n for n > 3.

Finally, take m + k = 2n. So (*) gives n + 2 = m(2n – m). Again we may take m ≥ k. m = 2n – 1 is not a solution (we are assuming n > 3). So 2n – m ≥ 2, and hence m(2n – m) ≥ 2m ≥ 2n > n + 2.

B2

x.

Answer

f(x) = –x/(x+1).

Solution

Suppose f(a) = a. Then putting x = y = a in the relation given, we get f(b) = b, where $b = 2a + a^2$. If –1 < a < 0, then –1 < b < a. But f(a)/a = f(b)/b. Contradiction. Similarly, if a > 0, then b > a, but f(a)/a = f(b)/b. Contradiction. So we must have a = 0.

But putting x = y in the relation given we get f(k) = k for k = x + f(x) + xf(x). Hence for any x we have x + f(x) + xf(x) = 0 and hence f(x) = –x/(x+1).

Finally, it is straightforward to check that f(x) = –x(x+1) satisfies the two conditions.

Thanks to Gerhard Woeginger for pointing out the error in the original solution and supplying this solution.

B3

Let the primes be $p_1 < p_2 < p_3 < \dots$. Let A consists of all products of n distinct primes such that the smallest is greater than p_n. For example: all primes except 2 are in A; 21 is not in A because it is a product of two distinct primes and the smallest is greater than 3. Now let $S = \{p_{i1}, p_{i2}, \dots \}$ be any infinite set of primes. Assume that $p_{i1} < p_{i2} < \dots$. Let $n = i_1$. Then $p_{i1}p_{i2} \dots p_{in}$ is not in A because it is a product of n distinct primes, but the smallest is not greater than p_n. But $p_{i2}p_{i3} \dots p_{in+1}$ is in A, because it is a product of n distinct primes and the smallest is greater than p_n. But both numbers are products of n distinct elements of S.

IMO 1995

A1

Let DN meet XY at Q. Angle QDZ = 90^o – angle NBD = angle BPZ. So triangles QDZ and BPZ are similar. Hence QZ/DZ = BZ/PZ, or QZ = BZ·DZ/PZ. Let AM meet XY at Q'. Then the same argument shows that Q'Z = AZ·CZ/PZ. But BZ·DZ = XZ·YZ = AZ·CZ, so QZ = Q'Z. Hence Q and Q' coincide.

A2

Put a = 1/x, b = 1/y, c = 1/z. Then $1/(a^3(b+c)) = x^3yz/(y+z) = x^2/(y+z)$. Let the expression given be E. Then by Cauchy's inequality we have (y+z + z+x + x+y)E $\geq (x + y + z)^2$, so E $\geq$ (x + y + z)/2. But applying the arithmetic/geometric mean result to x, y, z gives (x + y + z) $\geq$ 3. Hence result.

Thanks to Gerhard Woeginger for pointing out that the original solution was wrong.

A3

Answer

n = 4.

Solution

The first point to notice is that if no arrangement is possible for n, then no arrangement is possible for any higher integer. Clearly the four points of a square work for n = 4, so we focus on n = 5.

If the 5 points form a convex pentagon, then considering the quadrilateral $A_1A_2A_3A_4$ as made up of two triangles in two ways, we have that $r_1 + r_3 = r_2 + r_4$. Similarly, $A_5A_1A_2A_3$ gives $r_1 + r_3 = r_2 + r_5$, so $r_4 = r_5$.

We show that we cannot have two r's equal (whether or not the 4 points form a convex pentagon). For suppose $r_4 = r_5$. Then $A_1A_2A_4$ and $A_1A_2A_5$ have equal area. If A_4 and A_5 are on the same side of the line A_1A_2, then since they must be equal distances from it, A_4A_5 is parallel to A_1A_2. If they are on opposite sides, then the midpoint of A_4A_5 must lie on A_1A_2. The same argument can be applied to A_1 and A_3, and to A_2 and A_3. But we cannot have two of A_1A_2, A_1A_3 and A_2A_3 parallel to A_4A_5, because then A_1, A_2 and A_3 would be collinear. We also cannot have the midpoint of A_4A_5 lying on two of A_1A_2, A_1A_3 and A_2A_3 for the same reason. So we have established a contradiction. hence no two of the r's can be equal. In particular, this shows that the 5 points cannot form a convex pentagon.

Suppose the convex hull is a quadrilateral. Without loss of generality, we may take it to be $A_1A_2A_3A_4$. A_5 must lie inside one of $A_1A_2A_4$ and $A_2A_3A_4$. Again without loss of generality we may take it to be the latter, so that $A_1A_2A_5A_4$ is also a convex quadrilateral. Then $r_2 + r_4 = r_1 + r_3$ and also $= r_1 + r_5$. So $r_3 = r_5$, giving a contradiction as before.

The final case is the convex hull a triangle, which we may suppose to be $A_1A_2A_3$. Each of the other two points divides its area into three triangles, so we have: $(r_1 + r_2 + r_4) + (r_2 + r_3 + r_4) + (r_3 + r_1 + r_4) = (r_1 + r_2 + r_5) + (r_2 + r_3 + r_5) + (r_3 + r_1 + r_5)$ and hence $r_4 = r_5$, giving a contradiction.

So the arrangement is not possible for 5 and hence not for any $n > 5$.

B1

Answer

2^{997}.

Solution

The relation given is a quadratic in x_i, so it has two solutions, and by inspection these are $x_i = 1/x_{i-1}$ and $x_{i-1}/2$. For an even number of moves we can start with an arbitrary x_0 and get back to it. Use $n-1$ halvings, then take the inverse, that gets to $2^{n-1}/x_0$ after n moves. Repeating brings you back to x_0 after 2n moves. However, 1995 is odd!

The sequence given above brings us back to x_0 after n moves, provided that $x_0 = 2^{(n-1)/2}$. We show that this is the largest possible x_0. Suppose we have a halvings followed by an inverse followed by b halvings followed by an inverse. Then if the number of inverses is odd we end up with $2^{a-b+c-\cdots}/x_0$, and if it is even we end up with $x_0/2^{a-b+c-\cdots}$. In the first case, since the final number is x_0 we must have $x_0 = 2^{(a-b+-\ldots)/2}$. All the a, b, ... are non-negative and sum to the number of moves less the number of inverses, so we clearly maximise x_0 by taking a single inverse and $a = n-1$. In the second case, we must have $2^{a-b+c-\cdots} = 1$ and hence $a - b + c - \ldots = 0$. But that implies that $a + b + c + \ldots$ is even and hence the total number of moves is even, which it is not. So we must have an odd number of inverses and the maximum value of x_0 is $2^{(n-1)/2}$.

B2

BCD is an equilateral triangle and AEF is an equilateral triangle. The presence of equilateral triangles and quadrilaterals suggests using Ptolemy's inequality. [If this is unfamiliar, see ASU 61/6 solution.]. From CBGD, we get $CG\cdot BD \le BG\cdot CD + GD\cdot CB$, so $CG \le BG + GD$. Similarly from HAFE we get $HF \le HA + HE$. Also CF is shorter than the indirect path C to G to H to F, so $CF \le CG + GH + HF$. But we do not get quite what we want.

However, a slight modification of the argument does work. BAED is symmetrical about BE (because BA = BD and EA = ED). So we may take C' the reflection of C in the line BE and F' the reflection of F. Now C'AB and F'ED are still equilateral, so the same argument gives C'G ≥ AG + GB and HF' ≤ DH + HE. So CF = C'F' ≤ C'G + GH + HF' ≤ AG + GB + GH + DH + HE.

B3

Answer

2 + (2pCp – 2)/p, where 2pCp is the binomial coefficient (2p)!/(p!p!).

Solution

Let A be a subset other than {1, 2, ... , p} and {p+1, p+2, ... , 2p}. Consider the elements of A in {1, 2, ... , p}. The number r satisfies 0 < r < p. We can change these elements to another set of r elements of {1, 2, ... , p} by adding 1 to each element (and reducing mod p if necessary). We can repeat this process and get p sets in all. For example, if p = 7 and the original subset of {1, 2, ... , 7} was {3 , 5}, we get:

{3 , 5}, {4, 6}, {5, 7}, {6, 1}, {7, 2}, {1, 3}, {2, 4}.

The sum of the elements in the set is increased by r each time. So, since p is prime, the sums must form a complete set of residues mod p. In particular, they must all be distinct and hence all the subsets must be different.

Now consider the sets A which have a given intersection with {p+1, ... , n}. Suppose the elements in this intersection sum to k mod p. The sets can be partitioned into groups of p by the process described above, so that exactly one member of each group will have the sum -k mod p for its elements in {1, 2, ... , p}. In other words, exactly one member of each group will have the sum of all its elements divisible by p.

There are 2pCp subsets of {1, 2, ... , 2p} of size p. Excluding {1, 2, ... , p} and {p+1, ... , 2p} leaves (2pCp – 2). We have just shown that (2pCp – 2)/p of these have sum divisible by p. The two excluded subsets also have sum divisible by p, so there are 2 + (2pCp – 2)/p subsets in all having sum divisible by p.

IMO 1996

A1

Answer

No.

Solution

(a) Suppose the move is a units in one direction and b in the orthogonal direction. So $a^2 + b^2 = r$. If r is divisible by 2, then a and b are both even or both odd. But that means that we can only access the black squares or the white squares (assuming the rectangle is colored like a chessboard). The two corners are of opposite color, so the task cannot be done. All squares are congruent to 0 or 1 mod 3, so if r is divisible by 3, then a and b must both be multiples of 3. That means that if the starting square has coordinates (0,0), we can only move to squares of the form (3m,3n). The required destination is (19,0) which is not of this form, so the task cannot be done.

(b) If r = 73, then we must have a = 8, b = 3 (or vice versa). There are 4 types of move:

A: (x,y) to (x+8,y+3)
B: (x,y) to (x+3,y+8)
C: (x,y) to (x+8,y−3)
D: (x,y) to (x+3,y−8)

We regard (x,y) to (x−8,y−3) as a negative move of type A, and so on. Then if we have a moves of type A, b of type B and so on, then we require:

$$8(a + c) + 3(b + d) = 19; \ 3(a - c) + 8(b - d) = 0.$$

A simple solution is a = 5, b = −1, c = −3, d = 2, so we start by looking for solutions of this type. After some fiddling we find:

(0,0) to (8,3) to (16,6) to (8,9) to (11,1) to (19,4) to (11,7) to (19,10) to (16,2) to (8,5) to (16,8) to (19,0).

(c) If r = 97, then we must have a = 9, b = 4. As before, assume we start at (0,0). A good deal of fiddling around fails to find a solution, so we look for reasons why one is impossible. Call moves which change y by 4 "toggle" moves. Consider the central strip y = 4, 5, 6 or 7. Toggle moves must toggle us in and out of the strip. Non-toggle moves cannot be made if we are in the strip and keep us out of it if we are out of it. Toggle moves also change the parity of the x-coordinate, whereas non-toggle moves do not. Now we start and finish out of the strip, so we need an even number of toggle moves. On the other hand, we start with even x and end with odd x, so we need an odd number of toggle moves. Hence the task is impossible.

A2

We need two general results: the angle bisector theorem; and the result about the feet of the perpendiculars from a general point inside a triangle. The second is not so well-known. Let P be a general point in the triangle ABC with X, Y, Z the feet of the perpendiculars to BC, CA, AB. Then PA = YZ/sin A and □APB – □C = □XZY. To prove the first part: AP = AY/sin APY = AY/sin AZY (since AYPZ is cyclic) = YZ/sin A (sine rule). To prove the second part: □XZY = □XZP + □YZP = □XBP + □YAP = 90^o – □XPB + 90^o – □YPA = 180^o – (360^o – □APB – □XPY) = -180^o + □APB + (180^o – □C) = □APB – □C.

So, returning to the problem, □APB – □C = □XZY and □APC – □B = □XYZ. Hence XYZ is isosceles: XY = XZ. Hence PC sin C = PB sin B. But AC sin C = AB sin B, so AB/PB = AC/PC. Let the angle bisector BD meet AP at W. Then, by the angle bisector theorem, AB/PB = AW/WP. Hence AW/WP = AC/PC, so, by the angle bisector theorem, CW is the bisector of angle ACP, as required.

A3

Setting m = n = 0, the given relation becomes: f(f(0)) = f(f(0)) + f(0). Hence f(0) = 0. Hence also f(f(0)) = 0. Setting m = 0, now gives f(f(n)) = f(n), so we may write the original relation as f(m + f(n)) = f(m) + f(n).

So f(n) is a fixed point. Let k be the smallest non-zero fixed point. If k does not exist, then f(n) is zero for all n, which is a possible solution. If k does exist, then an easy induction shows that f(qk) = qk for all non-negative integers q. Now if n is another fixed point, write n = kq + r, with $0 \le r < k$. Then f(n) = f(r + f(kq)) = f(r) + f(kq) = kq + f(r). Hence f(r) = r, so r must be zero. Hence the fixed points are precisely the multiples of k.

But f(n) is a fixed point for any n, so f(n) is a multiple of k for any n. Let us take $n_1, n_2, \ldots , n_{k-1}$ to be arbitrary non-negative integers and set $n_0 = 0$. Then the most general function satisfying the conditions we have established so far is:

$$f(qk + r) = qk + n_r k \text{ for } 0 \le r < k.$$

We can check that this satisfies the functional equation. Let m = ak + r, n = bk + s, with $0 \le r, s < k$. Then f(f(m)) = f(m) = ak + n_rk, and f(n) = bk + n_sk, so f(m + f(n)) = ak + bk + n_rk + n_sk, and f(f(m)) + f(n) = ak + bk + n_rk + n_sk. So this is a solution and hence the most general solution.

B1

Answer

481^2.

Solution

Put 15a ± 16b = m^2, 16a – 15b = n^2. Then $15m^2 + 16n^2$ = 481a = 13·37a. The quadratic residues mod 13 are 0, ±1, ±3, ±4, so the residues of $15m^2$ are 0, ±2, ±5, ±6, and the residues of $16n^2$ are 0, ±1, ±3, ±4. Hence m and n must both be divisible by 13. Similarly, the quadratic residues of 37 are 0, ±1, ±3, ±4, ±7, ±9, ±10, ±11, ±12, ±16, so the residues of $15m^2$ are 0, ±2, ±5, ±6, ±8, ±13, ±14, ±15, ±17, ±18, and the residues of $16n^2$ are 0, ±1, ±3, ±4, ±7, ±9, ±10, ±11, ±12, ±16. Hence m and n must both be divisible by 37. Put m = 481m', n = 481n' and we get: $a = 481(15m'^2 + 16n'^2)$. We also have $481b = 16m^2 - 15n^2$ and hence $b = 481(16m'^2 - 15n'^2)$. The smallest possible solution would come from putting m' = n' = 1 and indeed that gives a solution.

This solution is straightforward, but something of a slog - all the residues have to be calculated. A more elegant variant is to notice that $m^4 + n^4 = 481(a^2 + b^2)$. Now if m and n are not divisible by 13 we have $m^4 + n^4 = 0 \pmod{13}$. Take k so that km = 1 (mod 13), then $(nk)^4 = -(mk)^4 = -1 \pmod{13}$. But that is impossible because then $(nk)^{12} = -1 \pmod{13}$, but $x^{12} = 1 \pmod{13}$ for all non-zero residues. Hence m and n are both divisible by 13. The same argument shows that m and n are both divisible by 37.

B2

The starting point is the formula for the circumradius R of a triangle ABC: 2R = a/sin A = b/sin B = c/sin C. [Proof: the side a subtends an angle 2A at the center, so a = 2R sin A.] This gives that $2R_A$ = BF/sin A, $2R_C$ = BD/sin C, $2R_E$ = FD/sin E. It is clearly not true in general that BF/sin A > BA + AF, although it is true if angle FAB $\geq 120^\circ$, so we need some argument that involves the hexagon as a whole.

Extend sides BC and FE and take lines perpendicular to them through A and D, thus forming a rectangle. Then BF is greater than or equal to the side through A and the side through D. We may find the length of the side through A by taking the projections of BA and AF giving AB sin B + AF sin F. Similarly the side through D is CD sin C + DE sin E. Hence:

2BF ≥ AB sin B + AF sin F + CD sin C + DE sin E. Similarly:

2BD ≥ BC sin B + CD sin D + AF sin A + EF sin E, and

2FD ≥ AB sin A + BC sin C + DE sin D + EF sin F.

Hence 2BF/sin A + 2BD/sin C + 2FD/sin E ≥ AB(sin A/sin E + sin B/sin A) + BC(sin B/sin C + sin C/sin E) + CD(sin C/sin A + sin D/sin C) + DE(sin E/sin A + sin D/sin E) + EF(sin E/sin C + sin F/sin E) + AF(sin F/sin A + sin A/sin C).

We now use the fact that opposite sides are parallel, which implies that opposite angles are equal: A = D, B = E, C = F. Each of the factors multiplying the sides in the last expression now has the form x + 1/x which has minimum value 2 when x = 1. Hence 2(BF/sin A + BD/sin C + FD/sin E) ≥ 2p and the result is proved.

B3

Let $x_i - x_{i-1} = p$ occur r times and $x_i - x_{i-1} = -q$ occur s times. Then r + s = n and pr = qs. If p and q have a common factor d, the $y_i = x_i/d$ form a similar set with p/d and q/d. If the result is true for the y_i then it must also be true for the x_i. So we can assume that p and q are relatively prime. Hence p divides s. Let s = kp. If k = 1, then p = s and q = r, so p + q = r + s = n. But we are given p + q < n. Hence k > 1. Let p + q = n/k = h.

Up to this point everything is fairly obvious and the result looks as though it should be easy, but I did not find it so. Some fiddling around with examples suggested that we seem to get $x_i = x_j$ for j = i + h. We observe first that $x_{i+h} - x_i$ must be a multiple of h. For suppose e differences are p, and hence h−e are −q. Then $x_{i+h} - x_i = ep - (h - e)q = (e - q)h$.

The next step is not obvious. Let $d_i = x_{i+h} - x_i$. We know that all d_is are multiples of h. We wish to show that at least one is zero. Now $d_{i+1} - d_i = (x_{i+h+1} - x_{i+h}) - (x_{i+1} - x_i)$ = (p or −q) − (p or −q) = 0, h or −h. So if neither of d_i nor d_{i+1} are zero, then either both are positive or both are negative (a jump from positive to negative would require a difference of at least 2h). Hence if none of the d_is are zero, then all of them are positive, or all of them are negative. But $d_0 + d_h + \dots + d_{kh}$ is a concertina sum with value $x_n - x_0 = 0$. So this subset of the d_is cannot all be positive or all negative. Hence at least one d_i is zero.

IMO 1997

A1

(a) If m and n are both even, then f(m,n) = 0. Let M be the midpoint of the hypoteneuse. The critical point is that M is a lattice point. If we rotate the triangle through 180 to give the other half of the rectangle, we find that its coloring is the same. Hence S_1 and S_2 for the triangle are each half their values for the rectangle. But the values for the rectangle are equal, so they must also be equal for the triangle and hence f(m,n) = 0.

If m and n are both odd, then the midpoint of the hypoteneuse is the center of a square and we may still find that the coloring of the two halves of the rectangle is the same. This time S_1 and S_2 differ by one for the rectangle, so f(m,n) = 1/2.

(b) The result is immediate from (a) for m and n of the same parity. The argument in (a) fails for m and n with opposite parity, because the two halves of the rectangle are oppositely colored. Let m be the odd side. Then if we extend the side length m by 1 we form a new triangle which contains the original triangle. But it has both sides even and hence $S_1 = S_2$. The area added is a triangle base 1 and height n, so area n/2. The worst case would be that all this area was the same color, in which case we would get f(m,n) = n/2. But n <= max(m,n), so this establishes the result.

(c) Intuitively, it is clear that if the hypoteneuse runs along the diagonal of a series of black squares, and we then extend one side, the extra area taken in will be mainly black. We need to make this rigorous. For the diagonal to run along the diagonal of black squares we must have n = m. It is easier to work out the white area added by extending a side. The white area takes the form of a series of triangles each similar to the new n+1 x n triangle. The biggest has sides 1 and n/(n+1). The next biggest has sides (n−1)/n and (n−1)/(n+1), the next biggest (n−2)/n and (n−2)/(n+1) and so on, down to the smallest which is 1/n by 1/(n+1). Hence the additional white area is $1/2\ (n/(n+1) + (n-1)^2/(n(n+1)) + (n-2)^2/(n(n+1)) + \dots + 1/(n(n+1))\ ) = 1/(2n(n+1))\ (n^2 + \dots + 1^2) = (2n+1)/12$. Hence the additional black area is n/2 − (2n+1)/12 = n/3 − 1/12 and the black excess in the additional area is n/6 − 1/6. If n is even, then f(n,n) = 0 for the original area, so for the new triangle f(n+1,n) = (n−1)/6 which is unbounded.

A2

Extend BV to meet the circle again at X, and extend CW to meet the circle again at Y. Then by symmetry (since the perpendicular bisectors pass through the center of the circle) AU = BX and AU = CY. Also arc AX = arc BU, and arc AY = arc UC. Hence arc XY = arc BC and so angle BYC = angle XBY and hence TY = TB. So AU = CY = CT + TY = CT + TB.

A3

Without loss of generality we may assume $x_1 + \dots + x_n = +1$. [If not just reverse the sign of every x_i.] For any given arrangement x_i we use *sum* to mean $x_1 + 2x_2 + 3x_3 + \dots + nx_n$. Now if we add together the sums for $x_1, x_2, \dots , x_n$ and the *reverse* $x_n, x_{n-1}, \dots , x_1$, we get $(n+1)(x_1 + \dots + x_n) = n+1$. So either we are home with the original arrangement or its reverse, or they have sums of opposite sign, one greater than (n+1)/2 and one less than −(n+1)/2.

A transposition changes the sum from ka + (k+1)b + other terms to kb + (k+1)a + other terms. Hence it changes the sum by |a − b| (where a, b are two of the x_i) which does not exceed n+1. Now we can get from the original arrangement to its reverse by a sequence of transpositions. Hence at some point in this sequence the sum must fall in the interval [−(n+1)/2, (n+1)/2] (because to get from a point below it to a point above it in a single step requires a jump of more than n+1). That point gives us the required permutation.

B2

Answer

(1,1), (16,2), (27,3).

Solution

Notice first that if we have $a^m = b^n$, then we must have $a = c^e$, $b = c^f$, for some c, where m=fd, n=ed and d is the greatest common divisor of m and n. [Proof: express a and b as products of primes in the usual way.]

In this case let d be the greatest common divisor of a and b^2, and put a = de, b^2 = df. Then for some c, $a = c^e$, $b = c^f$. Hence $f\ c^e = e\ c^{2f}$. We cannot have e = 2f, for then the c's cancel to give e = f. Contradiction. Suppose 2f > e, then $f = e\ c^{2f-e}$. Hence e = 1 and $f = c^{2f-1}$. If c = 1, then f = 1 and we have the solution a = b = 1. If $c \ge 2$, then $c^{2f-1} \ge 2^f > f$, so there are no solutions.

Finally, suppose 2f < e. Then $e = f\ c^{e-2f}$. Hence f = 1 and $e = c^{e-2}$. $c^{e-2} \ge 2^{e-2} \ge e$ for $e \ge 5$, so we must have e = 3 or 4 (e > 2f = 2). e = 3 gives the solution a = 27, b = 3. e = 4 gives the solution a = 16, b = 2.

B3

The key is to derive a recurrence relation for f(n) [not for $f(2^n)$]. If n is odd, then the sum must have a 1. In fact, there is a one-to-one correspondence between sums for n and sums for n-1. So:

$$f(2n+1) = f(2n)$$

Now consider n even. The same argument shows that there is a one-to-one correspondence between sums for n-1 and sums for n which have a 1. Sums which do not have a 1 are in one-to-one correspondence with sums for n/2 (just halve each term). So:

$$f(2n) = f(2n - 1) + f(n) = f(2n - 2) + f(n).$$

The upper limit is now almost immediate. First, the recurrence relations show that f is monotonic increasing. Now apply the second relation repeatedly to $f(2^{n+1)}$ to get:

$f(2^{n+1}) = f(2^{n+1} - 2^n) + f(2^n - 2^{n-1} + 1) + \dots + f(2^n - 1) + f(2^n) = f(2^n) + f(2^n - 1) + \dots + f(2^{n-1} + 1) + f(2^n)$ (*)

and hence $f(2^{n+1}) \ge (2^{n-1} + 1)f(2^n)$

We can now establish the upper limit by induction. It is false for n = 1 and 2, but almost true for n = 2, in that: $f(2^2) = 2^{22/2}$. Now if $f(2^n) \le 2^{n2/2}$, then the inequality just established shows that $f(2^{n+1}) < 2^n 2^{n2/2} < 2^{(n2+2n+1)/2} = 2^{(n+1)2/2}$, so it is true for n + 1. Hence it is true for all n > 2.

Applying the same idea to the lower limit does not work. We need something stronger. We may continue (*) inductively to obtain $f(2^{n+1}) = f(2^n) + f(2^n - 1) + \dots + f(3) + f(2) + f(1) + 1$. (**) We now use the following lemma:

$$f(1) + f(2) + \dots + f(2r) \geq 2r\, f(r)$$

We group the terms on the lhs into pairs and claim that $f(1) + f(2r) \geq f(2) + f(2r-1) \geq f(3) + f(2r-2) \geq \dots \geq f(r) + f(r+1)$. If k is even, then $f(k) = f(k+1)$ and $f(2r-k) = f(2r+1-k)$, so $f(k) + f(2r+1-k) = f(k+1) + f(2r-k)$. If k is odd, then $f(k+1) = f(k) + f((k+1)/2)$ and $f(2r+1-k) = f(2r-k) + f((2r-k+1)/2)$, but f is monotone so $f((k+1)/2) \leq f((2r+1-k)/2)$ and hence $f(k) + f(2r+1-k) \geq f(k+1) + f(2r-k)$, as required.

Applying the lemma to (**) gives $f(2^{n+1)} > 2^{n+1}f(2^{n-1})$. This is sufficient to prove the lower limit by induction. It is true for n = 1. Suppose it is true for n. Then $f(2^{n+1}) > 2^{n+1}2^{(n-1)2/4} = 2^{(n2-2n+1+4n+4)/4} > 2^{(n+1)2/4}$, so it is true for n+1.

IMO 1998

A1

Let AC and BD meet at X. Let H, K be the feet of the perpendiculars from P to AC, BD respectively. We wish to express the areas of ABP and CDP in terms of more tractable triangles. There are essentially two different configurations possible. In the first, we have area PAB = area ABX + area PAX + area PBX, and area PCD = area CDX − area PCX − area PDX. So if the areas being equal is equivalent to: area ABX − area CDX + area PAX + area PCX + area PBX + area PDX = 0. ABX and CDX are right-angled, so we may write their areas as AX·BX/2 and CX·DX/2. We may also put AX = AH − HX = AH − PK, BX = BK − PH, CX = CH + PK, DX = DK + PH. The other triangles combine in pairs to give area ACP + area BDP = (AC·PH + BD·PK)/2. This leads, after some cancellation to AH·BK = CH·DK. There is a similar configuration with the roles of AB and CD reversed.

The second configuration is area PAB = area ABX + area PAX − PBX, area PCD = area CDX + area PDX − area PCX. In this case AX = AH + PK, BX = BK − PH, CX = CH − PK, DX = DK + PH. But we end up with the same result: AH·BK = CH·DK.

Now if ABCD is cyclic, then it follows immediately that P is the center of the circumcircle and AH = CH, BK = DK. Hence the areas of PAB and PCD are equal.

Conversely, suppose the areas are equal. If PA > PC, then AH > CH. But since PA = PB and PC = PD (by construction), PB > PD, so BK > DK. So AH·BK >

CH·DK. Contradiction. So PA is not greater than PC. Similarly it cannot be less. Hence PA = PC. But that implies PA = PB = PC = PD, so ABCD is cyclic.

A2

Let us count the number N of triples (judge, judge, contestant) for which the two judges are distinct and rate the contestant the same. There are $b(b-1)/2$ pairs of judges in total and each pair rates at most k contestants the same, so $N \le kb(b-1)/2$.

Now consider a fixed contestant X and count the number of pairs of judges rating X the same. Suppose x judges pass X, then there are $x(x-1)/2$ pairs who pass X and $(b-x)(b-x-1)/2$ who fail X, so a total of $(x(x-1) + (b-x)(b-x-1))/2$ pairs rate X the same. But $(x(x-1) + (b-x)(b-x-1))/2 = (2x^2 - 2bx + b^2 - b)/2 = (x - b/2)^2 + b^2/4 - b/2 \ge b^2/4 - b/2 = (b - 1)^2/4 - 1/4$. But $(b - 1)^2/4$ is an integer (since b is odd), so the number of pairs rating X the same is at least $(b - 1)^2/4$. Hence $N \ge a (b - 1)^2/4$. Putting the two inequalities together gives $k/a \ge (b - 1)/2b$.

A3

Let $n = p_1^{a_1}\dots p_r^{a_r}$. Then $d(n) = (a_1 + 1)(a_2 + 1) \dots (a_r + 1)$, and $d(n^2) = (2a_1 + 1)(2a_2 + 1) \dots (2a_r + 1)$. So the a_i must be chosen so that $(2a_1 + 1)(2a_2 + 1) \dots (2a_r + 1) = k (a_1 + 1)(a_2 + 1) \dots (a_r + 1)$. Since all $(2a_i + 1)$ are odd, this clearly implies that k must be odd. We show that conversely, given any odd k, we can find a_i.

We use a form of induction on k. First, it is true for k = 1 (take n = 1). Second, we show that if it is true for k, then it is true for $2^mk - 1$. That is sufficient, since any odd number has the form $2^mk - 1$ for some smaller odd number k. Take $a_i = 2^i((2^m - 1)k - 1)$ for i = 0, 1, ... , m−1. Then $2a_i + 1 = 2^{i+1}(2^m - 1)k - (2^{i+1} - 1)$ and $a_i + 1 = 2^i(2^m - 1)k - (2^i - 1)$. So the product of the $(2a_i + 1)$'s divided by the product of the $(a_i + 1)$'s is $2^m(2^m - 1)k - (2^m - 1)$ divided by $(2^m - 1)k$, or $(2^mk - 1)/k$. Thus if we take these a_is together with those giving k, we get $2^mk - 1$, which completes the induction.

B1

Answer (a, b) = (11, 1), (49, 1) or $(7k^2, 7k)$.

Solution

If a < b, then $b \ge a + 1$, so $ab^2 + b + 7 > ab^2 + b \ge (a + 1)(ab + 1) = a^2b + a + ab \ge a^2b + a + b$. So there can be no solutions with a < b. Assume then that $a \ge b$.

Let k = the integer $(a^2b + a + b)/(ab^2 + b + 7)$. We have $(a/b + 1/b)(ab^2 + b + 7) = ab^2 + a + ab + 7a/b + 7/b + 1 > ab^2 + a + b$. So $k < a/b + 1/b$. Now if $b \geq 3$, then $(b - 7/b) > 0$ and hence $(a/b - 1/b)(ab^2 + b + 7) = ab^2 + a - a(b - 7/b) - 1 - 7/b < ab^2 + a < ab^2 + a + b$. Hence either b = 1 or 2 or $k > a/b - 1/b$.

If $a/b - 1/b < k < a/b + 1/b$, then $a - 1 < kb < a + 1$. Hence a = kb. This gives the solution $(a, b) = (7k^2, 7k)$.

It remains to consider b = 1 and 2. If b = 1, then a + 8 divides $a^2 + a + 1$ and hence also $a(a + 8) - (a^2 + a + 1) = 7a - 1$, and hence also $7(a + 8) - (7a - 1) = 57$. The only factors bigger than 8 are 19 and 57, so a = 11 or 49. It is easy to check that (a, b) = (11, 1) and (49, 1) are indeed solutions.

If b = 2, then 4a + 9 divides $2a^2 + a + 2$, and hence also $a(4a + 9) - 2(2a^2 + a + 2) = 7a - 4$, and hence also $7(4a + 9) - 4(7a - 4) = 79$. The only factor greater than 9 is 79, but that gives a = 35/2 which is not integral. Hence there are no solutions for b = 2.

A variant on this from Johannes Tang Lek Huo is as follows:

We have $ab^2 + b + 7$ divides $a(ab^2 + b + 7) - b(a^2b + a + b) = 7a - b^2$. If $7a = b^2$, then b must be a multiple of 7, so b = 7k for some k. Then $a = 7k^2$, and it is easy to check that this is a solution. We cannot have $7a < b^2$ for then $0 < b^2 - 7a < ab^2 < ab^2 + b + 7$. If 7a > b, then we must have $7a - b \leq ab^2 + b + 7 > ab^2$, so $7 > b^2$, so b = 1 or 2.

We can then continue as above.

B2

We show that $RI^2 + SI^2 - RS^2 > 0$. The result then follows from the cosine rule.

BI is perpendicular to MK and hence also to RS. So $IR^2 = BR^2 + BI^2$ and $IS^2 = BI^2 + BS^2$. Obviously RS = RB + BS, so $RS^2 = BR^2 + BS^2 + 2\ BR\cdot BS$. Hence $RI^2 + SI^2 - RS^2 = 2\ BI^2 - 2\ BR\cdot BS$. Consider the triangle BRS. The angles at B and M are 90 – B/2 and 90 – A/2, so the angle at R is 90 – C/2. Hence BR/BM = cos A/2/cos C/2 (using the sine rule). Similarly, considering the triangle BKS, BS/BK = cos C/2/cos A/2. So $BR\cdot BS = BM\cdot BK = BK^2$. Hence $RI^2 + SI^2 - RS^2 = 2(BI^2 - BK^2) = 2\ IK^2 > 0$.

B3

Answer

120

Solution

Let $f(1) = k$. Then $f(kt^2) = f(t)^2$ and $f(f(t)) = k^2t$. Also $f(kt)^2 = 1 \cdot f(kt)^2 = f(k^3t^2) = f(1^2f(f(kt^2))) = k^2f(kt^2) = k^2f(t)^2$. Hence $f(kt) = k\ f(t)$.

By an easy induction $k^nf(t^{n+1}) = f(t)^{n+1}$. But this implies that k divides f(t). For suppose the highest power of a prime p dividing k is $a > b$, the highest power of p dividing f(t). Then $a > b(1 + 1/n)$ for some integer n. But then $na > (n + 1)b$, so k^n does not divide $f(t)^{n+1}$. Contradiction.

Let $g(t) = f(t)/k$. Then $f(t^2f(s)) = f(t^2kg(s)) = k\ f(t^2g(s) = k^2g(t^2g(s))$, whilst $s\ f(t)^2 = k^2s\ f(t)^2$. So $g(t^2g(s)) = s\ g(t)^2$. Hence g is also a function satisfying the conditions which evidently has smaller values than f (for $k > 1$). It also satisfies $g(1) = 1$. Since we want the smallest possible value of f(1998) we may restrict attention to functions f satisfying $f(1) = 1$.

Thus we have $f(f(t) = t$ and $f(t^2) = f(t)^2$. Hence $f(st)^2 = f(s^2t^2) = f(s^2f(f(t^2))) = f(s)^2f(t^2) = f(s)^2f(t)^2$. So $f(st) = f(s)\ f(t)$.

Suppose p is a prime and $f(p) = m \cdot n$. Then $f(m)f(n) = f(mn) = f(f(p)) = p$, so one of f(m), f(n) = 1. But if $f(m) = 1$, then $m = f(f(m)) = f(1) = 1$. So f(p) is prime. If $f(p) = q$, then $f(q) = p$.

Now we may define f arbitarily on the primes subject only to the conditions that each f(prime) is prime and that if $f(p) = q$, then $f(q) = p$. For suppose that $s = p_1^{a_1} \ldots p_r^{a_r}$ and that $f(p_i) = q_i$. If t has any additional prime factors not included in the q_i, then we may add additional p's to the expression for s so that they are included (taking the additional a's to be zero). So suppose $t = q_1^{b_1} \ldots q_r^{b_r}$.Then $t^2f(s) = q_1^{2b_1+a_1} \ldots q_r^{2b_r+a_r}$ and hence $f(t^2f(s) = p_1^{2b_1+a_1} \ldots p_r^{2b_r+a_r} = s\ f(t)^2$.

We want the minimum possible value of f(1998). Now $1998 = 2.3^3.37$, so we achieve the minimum value by taking $f(2) = 3$, $f(3) = 2$, $f(37) = 5$ (and $f(37) = 5$). This gives $f(1998) = 3 \cdot 2^3 5 = 120$.

IMO 1999

A1

by Gerhard Woeginger

The possible sets are just the regular n-gons ($n > 2$).

Let $A_1, A_2, \ldots, A_k$ denote the vertices of the convex hull of S (and take indices mod k as necessary). We show first that these form a regular k-gon. A_{i+1} must lie on the perpendicular bisector of A_i and A_{i+2} (otherwise its reflection would lie outside the hull). Hence the sides are all equal. Similarly, A_{i+1} and A_{i+2} must be reflections in the perpendicular bisector of A_i and A_{i+3} (otherwise one of the reflections would lie outside the hull). Hence all the angles are equal.

Any axis of symmetry for S must also be an axis of symmetry for the A_i, and hence must pass through the center C of the regular k-gon. Suppose X is a point of S in the interior of k-gon. Then it must lie inside or on some triangle $A_iA_{i+1}C$. C must be the circumcenter of $A_iA_{i+1}X$ (since it lies on the three perpendicular bisectors, which must all be axes of symmetry of S), so X must lie on the circle center C, through A_i and A_{i+1}. But all points of the triangle $A_iA_{i+1}X$ lie strictly inside this circle, except $A_i$and A_{i+1}, so X cannot be in the interior of the k-gon.

A2

Answer Answer: C = 1/8. Equality iff two x_i are equal and the rest zero.

Solution

By a member of the Chinese team at the IMO - does anyone know who?

$(\sum x_i)^4 = (\sum x_i^2 + 2 \sum_{i<j} x_ix_j)^2 \ge 4 (\sum x_i^2) (2 \sum_{i<j} x_ix_j) = 8 \sum_{i<j} (x_ix_j \sum x_k^2) \ge 8 \sum_{i<j} x_ix_j(x_i^2 + x_j^2)$.

The second inequality is an equality only if n - 2 of the x_i are zero. So assume that $x_3 = x_4 = \dots = x_n = 0$. Then for the first inequality to be an equality we require that $(x_1^2 + x_2^2) = 2 x_1x_2$ and hence that $x_1 = x_2$. However, that is clearly also sufficient for equality.

Alternative solution by Gerhard Woeginger

Setting $x_1 = x_2 = 1$, $x_i = 0$ for $i > 2$ gives equality with C = 1/8, so, C cannot be smaller than 1/8.

We now use induction on n. For n = 2, the inequality with C = 1/8 is equivalent to $(x_1 - x_2)^4 \ge 0$, which is true, with equality iff $x_1 = x_2$. So the result is true for n = 2.

For n > 2, we may take $x_1 + \dots + x_n = 1$, and $x_1 \le x_2 \le \dots \le x_n$. Now replace x_1 and x_2 by 0 and $x_1 + x_2$. The sum on the rhs is unchanged and the sum on the lhs is increased by $(x_1 + x_2)^3 S - (x_1^3 + x_2^3) S - x_1x_2(x_1^2 + x_2^2)$, where $S = x_3 + x_4 + \dots + x_n$. But S is at least 1/3 (the critical case is n = 3, $x_i = 1/3$), so this is at least $x_1x_2(x_1 + x_2 - x_1^2 - x_2^2)$. This is strictly greater than 0 unless $x_1 = 0$ (when it equals 0), so the result follows by induction.

Comment. The first solution is elegant and shows clearly why the inequality is true. The second solution is more plodding, but uses an approach which is more general and can be applied in many other cases. At least with hindsight, the first solution is not as impossible to find as one might think. A little playing around soon uncovers the fact that one can get C = 1/8 with two x_i equal and the rest zero, and that this looks like the best possible. One just has to make *the jump to replacing ($x_i^2 + x_j^2$) by $\sum x_k^2$. The solution is then fairly clear. Of course, that does not detract from the contestant's achievement, because I and almost everyone else who has looked at the problem failed to make that jump.*

A3

Answer

Answer: n/2 (n/2 + 1) = n(n + 2)/4.

Solution

Let n = 2m. Color alternate squares black and white (like a chess board). It is sufficient to show that m(m+1)/2 white squares are necessary and sufficient to deal with all the black squares.

This is almost obvious if we look at the diagonals.

Look first at the odd-length white diagonals. In every other such diagonal, mark alternate squares (starting from the border each time, so that r+1 squares are marked in a diagonal length 2r+1). Now each black diagonal is adjacent to a picked white diagonal and hence each black square on it is adjacent to a marked white square. In all 1 + 3 + 5 + ... + m−1 + m + m−2 + ... + 4 + 2 = 1 + 2 + 3 + ... + m = m(m+1)/2 white squares are marked. This proves sufficiency.

For necessity consider the alternate odd-length black diagonals. Rearranging, these have lengths 1, 3, 5, ... , 2m−1. A white square is only adjacent to squares in one of these alternate diagonals and is adjacent to at most 2 squares in it. So we need at least 1 + 2 + 3 + ... + m = m(m+1)/2 white squares.

B1

Answer

(1, p) for any prime p; (2, 2); (3, 3).

Solution

by Gerhard Woeginger, Technical University, Graz

Answer: (1, p) for any prime p; (2, 2); (3, 3).

Evidently (1, p) is a solution for every prime p. Assume n > 1 and take q to be the smallest prime divisor of n. We show first that q = p.

Let x be the smallest positive integer for which $(p - 1)^x = - 1 \pmod q$, and y the smallest positive integer for which $(p - 1)^y = 1 \pmod q$. Certainly y exists and indeed $y < q$, since $(p - 1)^{q-1} = 1 \pmod q$. We know that $(p - 1)^n = -1 \pmod q$, so x exists also. Writing $n = sy + r$, with $0 \le r < y$, we conclude that $(p - 1)^r = -1 \pmod q$, and hence $x \le r < y$ (r cannot be zero, since 1 is not $-1 \pmod q$).

Now write $n = hx + k$ with $0 \le k < x$. Then $-1 = (p - 1)^n = (-1)^h(p - 1)^k \pmod q$. h cannot be even, because then $(p - 1)^k = -1 \pmod q$, contradicting the minimality of x. So h is odd and hence $(p - 1)^k = 1 \pmod q$ with $0 \le k < x < y$. This contradicts the minimality of y unless $k = 0$, so $n = hx$. But $x < q$, so $x = 1$. So $(p - 1) = -1 \pmod q$. p and q are primes, so $q = p$, as claimed.

So p is the smallest prime divisor of n. We are also given that $n \le 2p$. So either $p = n$, or $p = 2$, $n = 4$. The latter does not work, so we have shown that $n = p$. Evidently $n = p = 2$ and $n = p = 3$ work. Assume now that $p > 3$. We show that there are no solutions of this type.

Expand $(p - 1)^p + 1$ by the binomial theorem, to get (since $(-1)^p = -1$): $1 + -1 + p^2 - 1/2\, p(p - 1)p^2 + p(p - 1)(p - 2)/6\, p^3 - \dots$
The terms of the form (bin coeff) p^i with $i >= 3$ are obviously divisible by p^3, since the binomial coefficients are all integral. Hence the sum is p^2 + a multiple of p^3. So the sum is not divisible by p^3. But for $p > 3$, p^{p-1} is divisible by p^3, so it cannot divide $(p - 1)^p + 1$, and there are no more solutions.

B2

Solution by Jean-Pierre Ehrmann

Let O, O_1, O_2 and r, r_1, r_2 be the centers and radii of C, C_1, C_2 respectively. Let EF meet the line O_1O_2 at W, and let $O_2W = x$. We need to prove that $x = r_2$.

Take rectangular coordinates with origin O_2, x-axis O_2O_1, and let O have coordinates (a, b). Notice that O and M do not, in general, lie on O_1O_2. Let AB meet the line O_1O_2 at V.

We observe first that $O_2V = r_2^2/(2 r_1)$. [For example, let X be a point of intersection of C_1 and C_2 and let Y be the midpoint of O_2X. Then O_1YO_2 and XVO_2 are similar. Hence, $O_2V/O_2X = O_2Y/O_2O_1$.]

An expansion (or, to be technical, a *homothecy*) center M, factor r/r_1 takes O_1 to O and EF to AB. Hence EF is perpendicular to O_1O_2. Also the distance of O_1 from EF is r_1/r times the distance of O from AB, so $(r_1 - x) = r_1/r\, (a - r_2^2/(2 r_1))$ (*).

We now need to find a. We can get two equations for a and b by looking at the distances of O from O_1 and O_2. We have:
$(r - r_1)^2 = (r_1 - a)^2 + b^2$, and
$(r - r_2)^2 = a^2 + b^2$.

Subtracting to eliminate b, we get $a = r_2^2/(2 r_1) + r - r r_2/r_1$. Substituting back in (*), we get $x = r_2$, as required.

Alternative solution by Marcin Kuczma, communicated Arne Smeets

Let C_1 and C_2 meet at X and Y, and let AN meet C_2 again at D. Then AE·AM = AX·AY = AD·AN, so triangles AED and ANM are similar. Hence □ADE = □AMN.

Take the tangent AP as shown. Then □PAN = □AMN = □ADE, so AP and DE are parallel. The homothecy center M mapping C to C_1 takes the line AP to the line ED, so ED is tangent to C_1 at E. A similar argument show that it is tangent to C_2 at D. The homothecy takes AB to EF, so EF is perpendicular to O_1O_2 (the line of centers). Hence O_2EF is isosceles. So angle O_2EF = angle O_2FE = angle DEO_2 (DE tangent). In other words, O_2E bisects angle DEF. But ED is tangent to C_2, so EF is also.

B3

Solution communicated by Ong Shien Jin

Let c = f(0) and A be the image f(R). If a is in A, then it is straightforward to find f(a): putting a = f(y) and x = a, we get $f(a - a) = f(a) + a^2 + f(a) - 1$, so $f(a) = (1 + c)/2 - a^2/2$ (*).

The next step is to show that A - A = R. Note first that c cannot be zero, for if it were, then putting y = 0, we get: f(x - c) = f(c) + xc + f(x) - 1 (**) and hence f(0) = f(c) = 1. Contradiction. But (**) also shows that f(x - c) - f(x) = xc + (f(c) - 1). Here x is free to vary over R, so xc + (f(c) - 1) can take any value in R.

Thus given any x in R, we may find a, b in A such that x = a - b. Hence f(x) = f(a - b) = f(b) + ab + f(a) - 1. So, using (*): $f(x) = c - b^2/2 + ab - a^2/2 = c - x^2/2$.

In particular, this is true for x in A. Comparing with (*) we deduce that c = 1. So for all x in R we must have $f(x) = 1 - x^2/2$. Finally, it is easy to check that this satisfies the original relation and hence is the unique solution.

IMO 2000

A1

Angle EBA = angle BDM (because CD is parallel to AB) = angle ABM (because AB is tangent at B). So AB bisects EBM. Similarly, BA bisects angle EAM. Hence E is the reflection of M in AB. So EM is perpendicular to AB and hence to CD. So it suffices to show that MP = MQ.

Let the ray NM meet AB at X. XA is a tangent so $XA^2 = XM \cdot XN$. Similarly, XB is a tangent, so $XB^2 = XM \cdot XN$. Hence XA = XB. But AB and PQ are parallel, so MP = MQ.

A2

An elegant solution due to Robin Chapman is as follows:

$(B - 1 + 1/C) = B(1 - 1/B + 1/(BC)) = B(1 + A - 1/B)$. Hence, $(A - 1 + 1/B)(B - 1 + 1/C) = B(A^2 - (1 - 1/B)^2) \le B A^2$. So the square of the product of all three $\le B A^2 C B^2 A C^2 = 1$.

Actually, that is not quite true. The last sentence would not follow if we had some negative left hand sides, because then we could not multiply the inequalities. But it is easy to deal separately with the case where $(A - 1 + 1/B)$, $(B - 1 + 1/C)$, $(C - 1 + 1/A)$ are not all positive. If one of the three terms is negative, then the other two must be positive. For example, if $A - 1 + 1/B < 0$, then $A < 1$, so $C - 1 + 1/A > 0$, and $B > 1$, so $B - 1 + 1/C > 0$. But if one term is negative and two are positive, then their product is negative and hence less than 1.

Few people would manage this under exam conditions, but there are plenty of longer and easier to find solutions!

A3

Answer

$k \ge 1/(N-1)$.

Solution

An elegant solution by Gerhard Woeginger is as follows:

Suppose $k < 1/(N-1)$, so that $k_0 = 1/k - (N - 1) > 0$. Let X be the sum of the distances of the points from the rightmost point. If a move does not change the rightmost point, then it reduces X. If it moves the rightmost point a distance z to the right, then it reduces X by at least $z/k - (N-1)z = k_0 z$. X cannot be reduced below nil. So the total distance moved by the rightmost point is at most X_0/k_0, where X_0 is the initial value of X.

Conversely, suppose $k \ge 1/(N-1)$, so that $k_1 = (N-1) - 1/k \ge 0$. We always move the leftmost point. This has the effect of moving the rightmost point $z > 0$ and increasing X by $(N-1)z - z/k = k_1 z \ge 0$. So X is never decreased. But $z \ge k X/(N-1) \ge k X_0/(N-1) > 0$. So we can move the rightmost point arbitrarily far to the right (and hence all the points, since another N−1 moves will move the other points to the right of the rightmost point).

B1

Answer

12. Place 1, 2, 3 in different boxes (6 possibilities) and then place n in the same box as its residue mod 3. Or place 1 and 100 in different boxes and 2 – 99 in the third box (6 possibilities).

Solution

An elegant solution communicated (in outline) by both Mohd Suhaimi Ramly and Fokko J van de Bult is as follows:

Let H_n be the corresponding result that for cards numbered 1 to n the only solutions are by residue mod 3, or 1 and n in separate boxes and 2 to n – 1 in the third box. It is easy to check that they *are* solutions. H_n is the assertion that there are no others. H_3 is obviously true (although the two cases coincide). We now use induction on n. So suppose that the result is true for n and consider the case n + 1.

Suppose n + 1 is alone in its box. If 1 is not also alone, then let N be the sum of the largest cards in each of the boxes not containing n + 1. Since $n + 2 \le N \le n + (n - 1) = 2n - 1$, we can achieve the same sum N as from a different pair of boxes as (n + 1) + (N – n – 1). Contradiction. So 1 must be alone and we have one of the solutions envisaged in H_{n+1}.

If n + 1 is not alone, then if we remove it, we must have a solution for n. But that solution cannot be the n, 1, 2 to n – 1 solution. For we can easily check that none of the three boxes will then accomodate n + 1. So it must be the mod 3 solution. We can easily check that in this case n + 1 must go in the box with matching residue, which makes the (n + 1) solution the other solution envisaged by H_{n+1}. That completes the induction.

My much more plodding solution (which I was quite pleased with until I saw the more elegant solution above) follows. It took about half-an-hour and shows the kind of kludge one is likely to come up with under time pressure in an exam!

With a suitable labeling of the boxes as A, B, C, there are 4 cases to consider:

Case 1: A contains 1; B contains 2; C contains 3
Case 2: A contains 1,2
Case 3: A contains 1, 3; B contains 2
Case 4: A contains 1; B contains 2, 3.

We show that Cases 1 and 4 each yield just one possible arrangement and Cases 2 and 3 none.

In Case 1, it is an easy induction that n must be placed in the same box as its residue (in other words numbers with residue 1 mod 3 go into A, numbers with residue 2 go into B, and numbers with residue 0 go into C). For (n + 1) + (n − 2) = n + (n − 1). Hence n + 1 must go in the same box as n − 2 (if they were in different boxes, then we would have two pairs from different pairs of boxes with the same sum). It is also clear that this is a possible arrangement. Given the sum of two numbers from different boxes, take its residue mod 3. A residue of 0 indicates that the third (unused) box was C, a residue of 1 indicates that the third box was A, and a residue of 2 indicates that the third box was B. Note that this unique arrangement gives 6 ways for the question, because there are 6 ways of arranging 1, 2 and 3 in the given boxes.

In Case 2, let n be the smallest number not in box A. Suppose it is in box B. Let m be the smallest number in the third box, C. m − 1 cannot be in C. If it is in A, then m + (n − 1) = (m − 1) + n. Contradiction (m is in C, n − 1 is in A, so that pair identifies B as the third box, but m − 1 is in A and n is in B, identifying C). So m − 1 must be in B. But (m − 1) + 2 = m + 1. Contradiction. So Case 2 is not possible.

In Case 3, let n be the smallest number in box C, so n − 1 must be in A or B. If n − 1 is in A, then (n − 1) + 2 = n + 2. Contradiction (a sum of numbers in A and B equals a sum from C and A). If n − 1 is in B, then (n − 1) + 3 = n + 2. Contradiction (a sum from B and A equals a sum from C and B). So Case 3 is not possible.

In Case 4, let n be the smallest number in box C. n − 1 cannot be in A, or (n − 1) + 2 = 3 + n (pair from A, B with same sum as pair from B, C), so n − 1 must be in B. Now n + 1 cannot be in A (or (n + 1) + 2 = 3 + n), or in B or C (or 1 + (n + 1) = 2 + n). So n + 1 cannot exist and hence n = 100. It is now an easy induction that all of 4, 5, ... 98 must be in B. For given that m is in B, if m + 1 were in A, we would have 100 + m = 99 + (m + 1). But this arrangement (1 in A, 2 − 99 in B, 100 in C) is certainly possible: sums 3 − 100 identify C as the third box, sum 101 identifies B as the third box, and sums 102–199 identify A as the third box. Finally, as in Case 1, this unique arrangement corresponds to 6 ways of arranging the cards in the given boxes.

B2

Answer

Yes

Solution

Note that for b odd we have $2^{ab} + 1 = (2^a + 1)(2^{a(b-1)} - 2^{a(b-2)} + \dots + 1)$, and so $2^a + 1$ is a factor of $2^{ab} + 1$. It is sufficient therefore to find m such that (1) m has only a few distinct prime factors, (2) $2^m + 1$ has a large number of distinct prime factors, (3) m divides $2^m + 1$. For then we can take k, a product of enough distinct primes dividing $2^m + 1$ (but not m), so that km has exactly 2000 factors. Then km still divides $2^m + 1$ and hence $2^{km} + 1$.

The simplest case is where m has only one distinct prime factor p, in other words it is a power of p. But if p is a prime, then p divides $2^p - 2$, so the only p for which p divides $2^p + 1$ is 3. So the questions are whether $a_h = 2^m + 1$ is (1) divisible by $m = 3^h$ and (2) has a large number of distinct prime factors.

$a_{h+1} = a_h(2^{2m} - 2^m + 1)$, where $m = 3^h$. But $2^m = (a_h - 1)$, so $a_{h+1} = a_h(a_h^2 - 3 a_h + 3)$. Now $a_1 = 9$, so an easy induction shows that 3^{h+1} divides a_h, which answers (1) affirmatively. Also, since a_h is a factor of a_{h+1}, any prime dividing a_h also divides a_{h+1}. Put $a_h = 3^{h+1}b_h$. Then $b_{h+1} = b_h(3^{2h+1}b_h^2 - 3^{h+2}b_h + 1)$. Now $(3^{2h+1}b_h^2 - 3^{h+2}b_h + 1) > 1$, so it must have some prime factor $p > 1$. But p cannot be 3 or divide b_h (since $(3^{2h+1}b_h^2 - 3^{h+2}b_h + 1)$ is a multiple of $3b_h$ plus 1), so b_{h+1} has at least one prime factor $p > 3$ which does not divide b_h. So b_{h+1} has at least h distinct prime factors greater than 3, which answers (2) affirmatively. But that is all we need. We can take m in the first paragraph above to be 3^{2000}: (1) m has only one distinct prime factor, (2) $2^m + 1 = 3^{2001}\, b_{2000}$ has at least 1999 distinct prime factors other than 3, (3) m divides $2^m + 1$. Take k to be a product of 1999 distinct prime factors dividing b_{2000}. Then N = km is the required number with exactly 2000 distinct prime factors which divides $2^N + 1$.

B3

Let O be the centre of the incircle. Let the line parallel to A_1A_2 through L_2 meet the line A_2O at X. We will show that X is the reflection of K_2 in L_2L_3. Let A_1A_3 meet the line A_2O at B_2. Now A_2K_2 is perpendicular to K_2B_2 and OL_2 is perpendicular to L_2B_2, so $A_2K_2B_2$ and OL_2B_2 are similar. Hence $K_2L_2/L_2B_2 = A_2O/OB_2$. But OA_3 is the angle bisector in the triangle $A_2A_3B_2$, so $A_2O/OB_2 = A_2A_3/B_2A_3$.

Take B'_2 on the line A_2O such that $L_2B_2 = L_2B'_2$ (B'_2 is distinct from B_2 unless L_2B_2 is perpendicular to the line). Then angle $L_2B'_2X$ = angle $A_3B_2A_2$. Also, since L_2X is parallel to A_2A_1, angle $L_2XB'_2$ = angle $A_3A_2B_2$. So the triangles $L_2XB'_2$ and $A_3A_2B_2$ are similar. Hence $A_2A_3/B_2A_3 = XL_2/B_2'L_2 = XL_2/B_2L_2$ (since $B'_2L_2 = B_2L_2$).

Thus we have shown that $K_2L_2/L_2B_2 = XL_2/B_2L_2$ and hence that $K_2L_2 = XL_2$. L_2X is parallel to A_2A_1 so angle $A_2A_1A_3$ = angle A_1L_2X = angle L_2XK_2 + angle L_2K_2X = 2 angle L_2XK_2 (isosceles). So angle L_2XK_2 = 1/2 angle $A_2A_1A_3$ = angle A_2A_1O. L_2X and A_2A_1 are parallel, so K_2X and OA_1 are parallel. But OA_1 is perpendicular to L_2L_3, so K_2X is also perpendicular to L_2L_3 and hence X is the reflection of K_2 in L_2L_3.

Now the angle $K_3K_2A_1$ = angle $A_1A_2A_3$, because it is 90° – angle $K_3K_2A_2 = 90^\circ$ – angle $K_3A_3A_2$ ($A_2A_3K_2K_3$ is cyclic with A_2A_3 a diameter) = angle $A_1A_2A_3$. So the reflection of K_2K_3 in L_2L_3 is a line through X making an angle $A_1A_2A_3$ with L_2X, in other words, it is the line through X parallel to A_2A_3.

Let M_i be the reflection of L_i in A_iO. The angle M_2XL_2 = 2 angle OXL_2 = 2 angle A_1A_2O (since A_1A_2 is parallel to L_2X) = angle $A_1A_2A_3$, which is the angle betwee L_2X and A_2A_3. So M_2X is parallel to A_2A_3, in other words, M_2 lies on the reflection of K_2K_3 in L_2L_3.

If follows similarly that M_3 lies on the reflection. Similarly, the line M_1M_3 is the reflection of K_1K_3 in L_1L_3, and the line M_1M_2 is the reflection of K_1K_2 in L_1L_2 and hence the triangle formed by the intersections of the three reflections is just $M_1M_2M_3$.

IMO 2001

A1

Take D on the circumcircle with AD parallel to BC. Angle CBD = angle BCA, so angle ABD $\geq 30^o$. Hence angle AOD $\geq 60^o$. Let Z be the midpoint of AD and Y the midpoint of BC. Then AZ $\geq$ R/2, where R is the radius of the circumcircle. But AZ = YX (since AZYX is a rectangle).

Now O cannot coincide with Y (otherwise angle A would be 90^o and the triangle would not be acute-angled). So OX > YX $\geq$ R/2. But XC = YC - YX < R - YX $\leq$ R/2. So OX > XC.

Hence angle COX < angle OCX. Let CE be a diameter of the circle, so that angle OCX = angle ECB. But angle ECB = angle EAB and angle EAB + angle BAC = angle EAC = 90^o, since EC is a diameter. Hence angle COX + angle BAC < 90^o.

A2

A not particularly elegant, but fairly easy, solution is to use Cauchy: $(\sum xy)^2 \leq \sum x^2 \sum y^2$.

To get the inequality the right way around we need to take $x^2 = a/a'$ [to be precise, we are taking $x_1^2 = a/a'$, $x_2^2 = b/b'$, $x_3^2 = c/c'$.]. Take $y^2 = a\,a'$, so that xy = a. Then we get $\sum a/a' >= (\sum a)^2/\sum a\,a'$.

Evidently we need to apply Cauchy again to deal with $\sum a\,a'$. This time we want $\sum a\,a' \leq$ something. The obvious X=a, Y=a' does not work, but if we put $X=a^{1/2}$, $Y=a^{1/2}a'$, then we have $\sum a\,a' \leq (\sum a)^{1/2} (\sum a\,a'^2)^{1/2}$. So we get the required inequality provided that $(\sum a)^{3/2} \geq (\sum a\,a'^2)^{1/2}$ or $(\sum a)^3 \geq \sum a\,a'^2$.

Multiplying out, this is equivalent to: $3(ab^2 + ac^2 + ba^2 + bc^2 + ca^2 + cb^2) \geq 18abc$, or $a(b - c)^2 + b(c - a)^2 + c(a - b)^2 \geq 0$, which is clearly true.

A3

Notice first that the result is not true for a 20 x 20 array. Make 20 rectangles each 2 x 10, labelled 1, 2, ... , 20. Divide the 20 x 20 array into four quadrants (each 10 x 10). In each of the top left and bottom right quadrants, place 5 rectangles horizontally. In each of the other two quadrants, place 5 rectangles vertically. Now each row intersects 5 vertical rectangles and 1 horizontal. In other words, it contains just 6 different numbers. Similarly each column. But any given number is in either 10 rows and 2 columns or vice versa, so no number is in 3 rows and 3 columns. [None of this is necessary for the solution, but it helps to show what is going on.]

Returning to the 21 x 21 array, assume that an arrangement is possible with no integer in at least 3 rows and at least 3 columns. Color a cell white if its integer appears in 3 or more rows and black if its integer appears in only 1 or 2 rows. We count the white and black squares.

Each row has 21 cells and at most 6 different integers. 6 x 2 < 21, so every row includes an integer which appears 3 or more times and hence in at most 2 rows. Thus at most 5 different integers in the row appear in 3 or more rows. Each such integer can appear at most 2 times in the row, so there are at most 5 x 2 = 10 white cells in the row. This is true for every row, so there are at most 210 white cells in total.

Similarly, any given column has at most 6 different integers and hence at least one appears 3 or more times. So at most 5 different integers appear in 2 rows or less. Each such integer can occupy at most 2 cells in the column, so there are at most 5 x 2 = 10 black cells in the column. This is true for every column, so there are at most 210 black cells in total.

This gives a contradiction since 210 + 210 < 441.

Comment. This looks easy, but (like question 6) I found it curiously difficult (it took me well over 2 hours). For a while I could not see how to do better than a 12 x 12 array (with 2 rows of 1s, then 2 rows of 2s etc), which was disorienting. Then I got the argument almost right, but not quite right, which took more time.

The original question was phrased in terms of 21 boys and 21 girls in a competition with an unknown number of problems. Each boy, girl pair solved at least one problem. Each competitor solved at most 6 problems. One had to show that some problem was solved by at least 3 boys and at least 3 girls. The recasting in the terms above is almost immediate.

Equally, one can easily recast the solution above into the competition format. Take any boy B_0. At least one of the questions he attempts must be attempted by 3 or more girls (because he attempts at most 6 questions and there are more than 6x2 girls). Hence he attempts at most 5 questions which are only attempted by less than 3 girls. So at most 5 x 2 = 10 of the 21 pairs (B_0, G) attempt a question attempted by less than 3 girls. So at most 210 of the 441 pairs pairs (B, G) attempt such a question. Similarly, at most 210 pairs (B, G) attempt a question attempted by less than 3 boys. Hence at least 21 pairs (B, G) attempt a question attempted by 3 or more girls and 3 or more boys. So there must be at least one such question.

Note that the arguments above generalise immediately to show that in a 4N+1 by 4N+1 array with at most N+1 different integers in each row and column, there is some integer that appears in at least 3 rows and 3 columns, but this is not true for a 4N by 4N array.

B1

This is a simple application of the pigeon hole principle.

The sum of all m! distinct residues mod m! is not divisible by m! because m! is even (since m > 1). [The residues come in pairs a and m! – a, except for m!/2.].

However, the sum of all f(x) as x ranges over all m! permutations is 1/2 (m+1)! $\sum n_i$, which is divisible by m! (since m+1 is even). So at least one residue must occur more than once among the f(x).

B2

Answer

Answer: 80^o.

Solution

This is an inelegant solution, but I did get it fast! Without loss of generality we can take length AB = 1. Take angle ABY = x. Note that we can now solve the two triangles AXB and AYB. In particular, using the sine rule, BX = sin $30^o/\sin(150^o-2x)$, AY = $\sin x/\sin(120^o-x)$, YB = $\sin 60^o/\sin(120^o-x)$. So we have an equation for x.

Using the usual formula for sin(a + b) etc, and writing s = sin x, c = cos x, we get: $2\sqrt{3}\ s^2c - 4sc - 2\sqrt{3}\ c^3 + 2\sqrt{3}\ c^2 + 6sc - 2s - \sqrt{3} = 0$ or $-\sqrt{3}\ (4c^3 - 2c^2 - 2c + 1) = 2s(2c^2 - 3c + 1)$. This has a common factor 2c – 1. So c = 1/2 or $-\sqrt{3}\ (2c^2 - 1) = 2s(c - 1)$ (*).

$c = 1/2$ means $x = 60^o$ or angle $B = 120^o$. But in that case the sides opposite A and B are parallel and the triangle is degenerate (a case we assume is disallowed). So squaring (*) and using $s^2 = 1 - c^2$, we get: $16c^4 - 8c^3 - 12c^2 + 8c - 1 = 0$. This has another factor $2c - 1$. Dividing that out we get: $8c^3 - 6c + 1 = 0$. But we remember that $4c^3 - 3c = \cos 3x$, so we conclude that $\cos 3x = -1/2$. That gives $x = 40^o, 80^o, 160^o, 200^o, 280^o, 320^o$. But we require that $x < 60^o$ to avoid degeneracy. Hence the angle $B = 2x = 80^o$.

I subsequently found this geometric solution on the official Wolfram site (Wolfram was one of the sponsors of IMO 2001). I cannot say it is much easier, but at least it is geometric.

Extend AB to X' with BX' = BX. Extend AY to Z with YZ = YB. Then AZ = AY + YZ = AY + YB = AB + BX = AB + BX' = AX'. Angle A = 60^o, so AZX' is equilateral.

Use B also to denote the angle at B. Then angle YBX = B/2. Also angle BXX' + angle BX'X = B. The triangle is isosceles by construction, so angle BX'X = B/2. Hence angle XX'Z = 60^o – B/2. X lies on the bisector of A and AZ = AX', so XZ = XX'. Hence XZX' = 60^o – B/2 also. But angle Z = 60^o, so angle YZX = B/2 = angle YBX.

Now YZ = YB, so angle YZB = angle YBZ. Hence angle XZB = angle XBX (they are the difference of pairs of equal angles). If X does not lie on BZ, then we can conclude that XB = XZ.

In that case, since XZ = XX', we have XB = XX'. But already XB = BX' (by construction), so BXX' is equilateral and hence B/2 = 60^o. But then angle B + angle A = 180^o, so the triangle ABC is degenerate (with C at infinity), which we assume is disallowed. Hence X must lie on BZ, which means Z = C and angle B = 2 angle C. Hence angle B = 80, angle C = 40.

B3

Note first that KL+MN > KM+LN > KN+LM, because (KL+MN) – (KM+LN) = (K – N)(L – M) > 0 and (KM+LN) – (KN+LM) = (K – L)(M – N) > 0.

Multiplying out and rearranging, the relation in the question gives $K^2 - KM + M^2 = L^2 + LN + N^2$. Hence $(KM + LN)(L^2 + LN + N^2) = KM(L^2 + LN + N^2) + LN(K^2 - KM + M^2) = KML^2 + KMN^2 + K^2LN + LM^2N = (KL + MN)(KN + LM)$. In other words (KM + LN) divides (KL + MN)(KN + LM).

Now suppose KL + MN is prime. Since it greater than KM + LN, it can have no common factors with KM + LN. Hence KM + LN must divide the smaller integer KN + LM. Contradiction.

Comment. This looks easy, but in fact I found it curiously difficult. It is easy to go around in circles getting nowhere. Either I am getting older, or this is harder than it looks!

Note that it is not hard to find K, L, M, N satisfying the condition in the question. For example 11, 9, 5, 1.

IMO 2002

A1

Let a_i be the number of blue members (h, k) in S with h = i, and let b_i be the number of blue members (h, k) with k = i. It is sufficient to show that $b_0, b_1, \dots, b_{n-1}$ is a rearrangement of $a_0, a_1, \dots, a_{n-1}$ (because the number of type 1 subsets is the product of the a_i and the number of type 2 subsets is the product of the b_i).

Let c_i be the largest k such that (i, k) is red. If (i, k) is blue for all k then we put $c_i = -1$. Note that if $i < j$, then $c_i \geq c_j$, since if (j, c_i) is red, then so is (i, c_i). Note also that (i, k) is red for $k \leq c_i$, so the sequence $c_0, c_1, \dots, c_{n-1}$ completely defines the coloring of S.

Let S_i be the set with the sequence $c_0, c_1, \dots, c_i, -1, \dots, -1$, so that $S_{n-1} = S$. We also take S_{-1} as the set with the sequence $-1, -1, \dots, -1$, so that all its members are blue. We show that the rearrangement result is true for S_{-1} and that if it is true for S_i then it is true for S_{i+1}. It is obvious for S_{-1}, because both a_i and b_i are $n, n-1, \dots, 2, 1$. So suppose it is true for S_i (where $i < n-1$). The only difference between the a_j for S_i and for S_{i+1} is that $a_{i+1} = n-i-1$ for S_i and $(n-i-1)-(c_{i+1}+1)$ for S_{i+1}. In other words, the number $n-i-1$ is replaced by the number $n-i-c-2$, where $c = c_{i+1}$. The difference in the b_j is that 1 is deducted from each of $b_0, b_1, \dots, b_c$. But these numbers are just $n-i-1, n-i-1, n-i-2, \dots, n-i-c-1$. So the effect of deducting 1 from each is to replace $n-i-1$ by $n-i-c-2$, which is the same change as was made to the a_j. So the rearrangement result also holds for S_{i+1}. Hence it holds for S.

A2

F is equidistant from A and O. But OF = OA, so OFA is equilateral and hence angle AOF = 60°. Since angle AOC > 60°, F lies between A and C. Hence the ray CJ lies between CE and CF.

D is the midpoint of the arc AB, so angle DOB = ½ angle AOB = angle ACB. Hence DO is parallel to AC. But OJ is parallel to AD, so AJOD is a parallelogram. Hence AJ = OD. So AJ = AE = AF, so J lies on the opposite side of EF to A and hence on the same side as C. So J must lie inside the triangle CEF.

Also, since EF is the perpendicular bisector of AO, we have AE = AF = OE, so A is the center of the circle through E, F and J. Hence angle EFJ = ½ angle EAJ. But angle EAJ = angle EAC (same angle) = angle EFC. Hence J lies on the bisector of angle EFC.

Since EF is perpendicular to AO, A is the midpoint of the arc EF. Hence angle ACE = angle ACF, so J lies on the bisector of angle ECF. Hence J is the incenter.

Many thanks to Dirk Laurie for pointing out that the original version of this solution failed to show the relevance of angle AOC > 60°. According to the official marking scheme, one apparently lost a mark for failing to show J lies inside CEF.

A3

Answer: m = 5, n = 3.

Obviously m > n. Take polynomials q(x), r(x) with integer coefficients and with degree r(x) < n such that $x^m + x - 1 = q(x)\ (x^n + x^2 - 1) + r(x)$. Then $x^n + x^2 - 1$ divides r(x) for infinitely many positive integers x. But for sufficiently large x, $x^n + x^2 - 1 > r(x)$ since r(x) has smaller degree. So r(x) must be zero. So $x^m + x - 1$ factorises as $q(x)\ (x^n + x^2 - 1)$, where $q(x) = x^{m-n} + a_{m-n-1}x^{m-n-1} + \ldots + a_0$.

At this point I use an elegant approach provided by Jean-Pierre Ehrmann

We have $(x^m + x - 1) = x^{m-n}(x^n + x^2 - 1) + (1 - x)(x^{m-n+1} + x^{m-n} - 1)$, so $(x^n + x^2 - 1)$ must divide $(x^{m-n+1} + x^{m-n} - 1)$. So, in particular, $m \geq 2n-1$. Also $(x^n + x^2 - 1)$ must divide $(x^{m-n+1} + x^{m-n} - 1) - x^{m-2n+1}(x^n + x^2 - 1) = x^{m-n} - x^{m-2n+3} + x^{m-2n+1} - 1$ (*).

At this point there are several ways to go. The neatest is Bill Dubuque's:

(*) can be written as $x^{m-2n+3}(x^{n-3} - 1) + (x^{m-(2n-1)} - 1)$ which is < 0 for all x in (0, 1) unless n − 3 = 0 and m − (2n − 1) = 0. So unless n = 3, m = 5, it is has no roots in (0, 1). But $x^n + x^2 - 1$ (which divides it) has at least one becaause it is −1 at x = 0 and +1 at x = 1. So we must have n = 3, m = 5. It is easy to check that in this case we have an identity.

Two alternatives follow. Jean-Pierre Ehrmann continued:

If m = 2n−1, (*) is $x^{n-1} - x^2$. If n = 3, this is 0 and indeed we find m = 5, n = 3 gives an identity. If n > 3, then it is $x^2(x^{n-3} - 1)$. But this has no roots in the interval (0, 1), whereas $x^n + x^2 - 1$ has at least one (because it is −1 at x = 0 and +1 at x = 1), so $x^n + x^2 - 1$ cannot be a factor.

If $m > 2n-1$, then (*) has four terms and factorises as $(x - 1)(x^{m-n-1} + x^{m-n-2} + \ldots + x^{m-2n+3} + x^{m-2n} + x^{m-2n-1} + \ldots + 1)$. Again, this has no roots in the interval (0, 1), whereas $x^n + x^2 - 1$ has at least one, so $x^n + x^2 - 1$ cannot be a factor.

François Lo Jacomo, having got to $x^n + x^2 - 1$ divides $x^{m-n+1} + x^{m-n} - 1$ and looking at the case $m - n + 1 > n$, continues:

$x^n + x^2 - 1$ has a root r such that $0 < r < 1$ (because it is −1 at x = 0 and +1 at x = 1). So $r^n = 1 - r^2$. It must also be a root of $x^m + x - 1$, so $1 - r = r^m \le r^{2n} = (1 - r^2)^2$. Hence $(1 - r^2)^2 - (1 - r) = (1 - r)\, r\, (1 - r - r^2) \ge 0$, so $1 - r - r^2 \ge 0$. Hence $r^n = 1 - r^2 \ge r$, which is impossible.

Many thanks to Carlos Gustavo Moreira for patiently explaining why the brute force approach of calculating the coefficients of q(x), starting at the low end, is full of pitfalls. After several failed attempts, I have given up on it!

B1

$d_{k+1-m} <= n/m$. So $d < n^2(1/(1.2) + 1/(2.3) + 1/(3.4) + \ldots)$. The inequality is certainly strict because d has only finitely many terms. But $1/(1.2) + 1/(2.3) + 1/(3.4) + \ldots = (1/1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + \ldots = 1$. So $d < n^2$.

Obviously d divides n^2 for n prime. Suppose n is composite. Let p be the smallest prime dividing n. Then $d > n^2/p$. But the smallest divisor of n^2 apart from 1 is p, so if d divides n^2, then $d \le n^2/p$. So d cannot divide n^2 for n composite.

B2

Answer: there are three possible functions: (1) f(x) = 0 for all x; (2) f(x) = 1/2 for all x; or (3) $f(x) = x^2$.

Put x = y = 0, u = v, then 4 f(0) f(u) = 2 f(0). So either f(u) = 1/2 for all u, or f(0) = 0. f(u) = 1/2 for all u is certainly a solution. So assume f(0) = 0.

Putting y = v = 0, f(x) f(u) = f(xu) (*). In particular, taking x = u = 1, $f(1)^2 = f(1)$. So f(1) = 0 or 1. Suppose f(1) = 0. Putting x = y = 1, v = 0, we get 0 = 2f(u), so f(x) = 0 or all x. That is certainly a solution. So assume f(1) = 1.

Putting x = 0, u = v = 1 we get 2 f(y) = f(y) + f(−y), so f(−y) = f(y). So we need only consider f(x) for x positive. We show next that $f(r) = r^2$ for r rational. The first step is to show that $f(n) = n^2$ for n an integer. We use induction on n. It is true for n = 0 and 1. Suppose it is true for n−1 and n. Then putting x = n, y = u = v = 1, we get 2f(n) + 2 = f(n−1) + f(n+1), so $f(n+1) = 2n^2 + 2 - (n-1)^2 = (n+1)^2$ and it is true for n+1. Now (*) implies that f(n) f(m/n) = f(m), so $f(m/n) = m^2/n^2$ for integers m, n. So we have established $f(r) = r^2$ for all rational r.

From (*) above, we have $f(x^2) = f(x)^2 \geq 0$, so f(x) is always non-negative for positive x and hence for all x. Putting u = y, v = x, we get $(f(x) + f(y))^2 = f(x^2 + y^2)$, so $f(x^2 + y^2) = f(x)^2 + 2f(x)f(y) + f(y)^2 \geq f(x)^2 = f(x^2)$. For any u > v > 0, we may put $u = x^2 + y^2$, $v = x^2$ and hence $f(u) \geq f(v)$. In other words, f is an increasing function.

So for any x we may take a sequence of rationals r_n all less than x we converge to x and another sequence of rationals s_n all greater than x which converge to x. Then $r_n^2 = f(r_n) \leq f(x) \leq f(s_n) = s_n^2$ for all x and hence $f(x) = x^2$.

B3

Denote the circle center O_i by C_i. The tangents from O_1 to C_i contain an angle 2x where $\sin x = 1/O_1O_i$. So $2x > 2/O_1O_i$. These double sectors cannot overlap, so $\sum 2/O_1O_i < \pi$. Adding the equations derived from O_2, O_3, ... we get $4 \sum O_iO_j < n\pi$, so $\sum O_iO_j < n\pi/4$, which is not quite good enough.

There are two key observations. The first is that it is better to consider the angle $O_iO_1O_j$ than the angle between the tangents to a single circle. It is not hard to show that this angle must exceed both $2/O_1O_i$ and $2/O_1O_j$. For consider the two common tangents to C_1 and C_i which intersect at the midpoint of O_1O_i. The angle between the center line and one of the tangents is at least $2/O_1O_i$. No part of the circle C_j can cross this line, so its center O_j cannot cross the line parallel to the tangent through O_1. In other word, angle $O_iO_1O_j$ is at least $2/O_1O_i$. A similar argument establishes it is at least $2/O_1O_j$.

Now consider the convex hull of the n points O_i. $m \leq n$ of these points form the convex hull and the angles in the convex m-gon sum to $(m-2)\pi$. That is the second key observation. That gains us not one but two amounts $\pi/4$. However, we lose one back. Suppose O_1 is a vertex of the convex hull and that its angle is θ_1. Suppose for convenience that the rays O_1O_2, O_1O_3, ... , O_1O_n occur in that order with O_2 and O_n adjacent vertices to O_1 in the convex hull. We have that the n-2 angles between adjacent rays sum to θ_1. So we have $\sum 2/O_1O_i < \theta_1$, where the sum is taken over only n-2 of the i, not all n-1. But we can choose which i to drop, because of our freedom to choose either distance for each angle. So we drop the longest distance O_1O_i. [If O_1O_k is the longest, then we work outwards from that ray. Angle $O_{k-1}O_1O_k > 2/O_1O_{k-1}$, and angle $O_kO_1O_{k+1} > 2/O_1O_{k+1}$ and so on.]

We now sum over all the vertices in the convex hull. For any centers O_i inside the hull we use the $\sum_j 2/O_iO_j < \pi$ which we established in the first paragraph, where the sum has all n-1 terms. Thus we get $\sum_{i,j} 2/O_iO_j < (n-2)\pi$, where for vertices i for which O_i is a vertex of the convex hull the sum is only over n-2 values of j and excludes $2/O_iO_{max\ i}$ where $O_{max\ i}$denotes the furthest center from O_i.

Now for O_i a vertex of the convex hull we have that the sum over all j, $\sum 2/O_iO_j$, is the sum Σ' over all but j = max i plus at most $1/(n-2)\ \Sigma'$. In other words we must increase the sum by at most a factor $(n-1)/(n-2)$ to include the missing term. For O_i not a vertex of the hull, obviously no increase is needed. Thus the full sum $\sum_{i,j} 2/O_iO_j < (n-1)\pi$. Hence $\sum_{i<j} 1/O_iO_j < (n-1)\pi/4$ as required.

IMO 2003

A1

Thanks to Li Yi

Having found $x_1, x_2, \dots, x_k$ there are $k\cdot 101\cdot 100$ forbidden values for x_{k+1} of the form $x_i + a_m - a_n$ with m and n unequal and another k forbidden values with m = n. Since $99\cdot 101\cdot 100 + 99 = 10^6 - 1$, we can successively choose 100 distinct x_i.

Gerhard Woeginger sent me a similar solution

A2

Answer

$(m, n) = (2k, 1), (k, 2k)$ or $(8k^4 - k, 2k)$

Solution

Thanks to Li Yi

The denominator is $2mn^2 - n^3 + 1 = n^2(2m - n) + 1$, so $2m >= n > 0$. If n = 1, then m must be even, in other words, we have the solution (m, n) = (2k, 1).

So assume n > 1. Put $h = m^2/(2mn^2 - n^3 + 1)$. Then we have a quadratic equation for m, namely $m^2 - 2hn^2m + (n^3 - 1)h = 0$. This has solutions hn^2 +− N, where N is the positive square root of $h^2n^4 - hn^3 + h$. Since n > 1, $h \geq 1$, N is certainly real. But the sum and product of the roots are both positive, so both roots must be positive. The sum is an integer, so if one root is a positive integer, then so is the other.

The larger root $hn^2 + N$ is greater than hn^2, so the smaller root $< h(n^3 - 1)/(hn^2) < n$. But note that if $2m - n > 0$, then since $h > 0$, we must have the denominator $(2m - n)n^2 + 1$ smaller than the numerator and hence m > n. So for the smaller root we cannot have $2m - n > 0$. But $2m - n$ must be non-negative (since h is positive), so $2m - n = 0$ for the smaller root. Hence $hn^2 - N = n/2$. Now $N^2 = (hn^2 - n/2)^2 = h^2n^4 - hn^3 + h$, so $h = n^2/4$. Thus n must be even. Put n = 2k and we get the solutions (m, n) = (k, 2k) and $(8k^4 - k, 2k)$.

We have shown that any solution must be of one of the three forms given, but it is trivial to check that they are all indeed solutions.

A3

Thanks to Li Yi

We use bold to denote vectors, so **AB** means the vector from A to B. We take some arbitrary origin and write the vector **OA** as **A** for short. Note that the vector to the midpoint of AB is $(\mathbf{A} + \mathbf{B})/2$, so the vector from the midpoint of DE to the midpoint of AB is $(\mathbf{A} + \mathbf{B} - \mathbf{D} - \mathbf{E})/2$. So the starting point is $|\mathbf{A} + \mathbf{B} - \mathbf{D} - \mathbf{E}| \geq \sqrt{3}\ (|\mathbf{A} - \mathbf{B}| + |\mathbf{D} - \mathbf{E}|)$ and two similar equations. The key is to notice that by the triangle inequality we have $|\mathbf{A} - \mathbf{B}| + |\mathbf{D} - \mathbf{E}| \geq |\mathbf{A} - \mathbf{B} - \mathbf{D} + \mathbf{E}|$ with equality iff the opposite sides AB and DE are parallel. Thus we get $|\mathbf{DA} + \mathbf{EB}| \geq \sqrt{3}\ |\mathbf{DA} - \mathbf{EB}|$. Note that DA and EB are diagonals. Squaring, we get $\mathbf{DA}^2 + 2\ \mathbf{DA}.\mathbf{EB} + \mathbf{EB}^2 \geq 3(\mathbf{DA}^2 - 2\ \mathbf{DA}.\mathbf{EB} + \mathbf{EB}^2)$, or $\mathbf{DA}^2 + \mathbf{EB}^2 \leq 4\ \mathbf{DA}.\mathbf{EB}$. Similarly, we get $\mathbf{EB}^2 + \mathbf{FC}^2 \leq 4\ \mathbf{EB}.\mathbf{FC}$ and $\mathbf{FC}^2 + \mathbf{AD}^2 \leq 4\ \mathbf{FC}.\mathbf{AD} = -4\ \mathbf{FC}.\mathbf{DA}$. Adding the three equations gives $2(\mathbf{DA} - \mathbf{EB} + \mathbf{FC})^2 \leq 0$. So it must be zero, and hence $\mathbf{DA} - \mathbf{EB} + \mathbf{FC} = 0$ and opposite sides of the hexagon are parallel.

Note that $\mathbf{DA} - \mathbf{EB} + \mathbf{FC} = \mathbf{A} - \mathbf{D} - \mathbf{B} + \mathbf{E} + \mathbf{C} - \mathbf{F} = \mathbf{BA} + \mathbf{DC} + \mathbf{FE}$. So $\mathbf{BA} + \mathbf{DC} + \mathbf{FE} = 0$. In other words, the three vectors can form a triangle.

Since EF is parallel to BC, if we translate EF along the vector **ED** we get CG, an extension of BC. Similarly, if we translate AB along the vector **BC** we get an extension of ED. Since **BA**, **DC** and **FE** form a triangle, AB must translate to DG. Thus HAB and CDG are congruent. Similarly, if we take AF and DE to intersect at I, the triangle FIE is also congruent (and similarly oriented) to HAB and CDG. Take J, K as the midpoints of AB, ED. HIG and HAB are equiangular and hence similar. IE = DG and K is the midpoint of ED, so K is also the midpoint of IG. Hence HJ is parallel to HK, so H, J, K are collinear.

Hence HJ/AB = HK/IG = (HK − HJ)/(IG − AB) = JK/(AB + ED) = $\frac{1}{2}\sqrt{3}$. Similarly, each of the medians of the triangle HAB is $\frac{1}{2}\sqrt{3}$ times the corresponding side. We will show that this implies it is equilateral. The required result then follows immediately.

Suppose a triangle has side lengths a, b, c and the length of the median to the midpoint of side length c is m. Then applying the cosine rule twice we get $m^2 = a^2/2 + b^2/2 - c^2/4$. So if $m^2 = \frac{3}{4} c^2$, it follows that $a^2 + b^2 = 2c^2$. Similarly, $b^2 + c^2 = 2a^2$. Subtracting, a = c. Similarly for the other pairs of sides.

An alternative (and rather more elegant) solution sent my some anonymous contestants at the IMO is as follows

Let the diagonals AD and BE meet at P. We show that angle APB $<= 60^\circ$. Suppose angle APB $> 60^\circ$. Take X and Y inside the hexagon so that ABX and

DEY are equilateral (as shown). Then since angle APB > angle AXB, P lies inside the circumcircle of ABX (which we take to have center O, radius r). Similarly, it lies inside the circumcircle of DEY (which we take to have center O' radius r'), so these circles must meet and hence OO' < r + r'. Now √3 (AB + DE)/2 = MN (where M, N are the midpoints of AB, DE) ≤ MO + OO' + O'N < r/2 + (r + r') + r'/2 = (3/2)(r + r') = √3 (AB + DE)/2. Contradiction.

The same argument applies to any two long diagonals. Hence the angles must all be 60^o. Also we must have MP ≤ MX with equality iff P = X, and similarly NP ≤ NY with equality iff P = Y. So MN ≤ MP + PN ≤ MX + NY = √3 (AB + DE)/2 = MN. Hence we have equality and so P = X = Y.

Hence angle APB = 60^o. Suppose AD and CF meet at Q. The same argument shows that angle AQF = 60^o. So the hexagon angle at A is angle APB + angle AQF = 120^o. Similarly for the other angles.

Finally, note that the only possible configuration is:

The ratio AB/BC is arbitrary, but the figure is symmetrical under rotations through 120^o. That follows immediately from either of the two solutions above.

B1

Thanks to Li Yi

APRD is cyclic with diameter AD (because angle APD = angle ARD = 90^o. Suppose its center is O and its radius r. Angle PAR = ½ angle POR, so PR = 2r sin ½POR = AD sin PAR. Similarly, RQ = CD sin RCQ. (Note that it makes no difference if R, P are on the same or opposite sides of the line AD.) But sin PAR = sin BAC, sin RCQ = sin ACB, so applying the sine rule to the triangle ABC, sin RCQ/sin PAR = AB/BC. Thus we have AD/CD = (PR/RQ) (AB/BC). Suppose the angle bisectors of B, D meet AD at X, Y. Then we have AB/BC = AX/CX and AD/CD = AY/CY. Hence (AY/CY)/(AX/CX) = PR/RQ. So PR = RQ iff X = Y, which is the required result.

Note that ABCD does not need to be cyclic! Exercise: does it need to be convex?

B2

Thanks to Li Yi

Notice first that if we restrict the sums to i < j, then they are halved. The lhs sum is squared and the rhs sum is not, so the the desired inequality with sums restricted to i < j has (1/3) on the rhs instead of (2/3).

Consider the sum of all $|x_i - x_j|$ with $i < j$. x_1 occurs in $(n-1)$ terms with a negative sign. x_2 occurs in one term with a positive sign and $(n-2)$ terms with a negative sign, and so on. So we get $-(n-1)x_1 - (n-3)x_2 - (n-5)x_3 - \ldots + (n-1)x_n = \sum (2i-1-n)x_i$.

We can now apply Cauchy-Schwartz. The square of this sum is just $\sum x_i^2 \sum (2i-1-n)^2$.

Looking at the other side of the desired inequality, we see immediately that it is $n \sum x_i^2 - (\sum x_i)^2$. We would like to get rid of the second term, but that is easy because if we add h to every x_i the sums in the desired inequality are unaffected (since they use only differences of x_i), so we can choose h so that $\sum x_i$ is zero. Thus we are home if we can show that $\sum (2i-1-n)^2 \le n(n^2 - 1)/3$. That is easy: lhs $= 4 \sum i^2 - 4(n+1) \sum i + n(n+1)^2 = (2/3)n(n+1)(2n+1) - 2n(n+1) + n(n+1)^2 = (1/3)n(n+1)(2(2n+1) - 6 + 3(n+1)) = (1/3)n(n^2 - 1) =$ rhs. That establishes the required inequality.

We have equality iff we have equality at the Cauchy-Schwartz step and hence iff x_i is proportional to $(2i-1-n)$. That implies that $x_{i+1} - x_i$ is constant. So equality implies that the sequence is an AP. But if the sequence is an AP with difference d (so $x_{i+1} = x_i + d$) and we take $x_1 = -(d/2)(n-1)$, then we get $x_i = (d/2)(2i-1-n)$ and $\sum x_i = 0$, so we have equality.

B3

Since $(p^p - 1)/(p-1) = 1 + p + p^2 + \cdots + p^{p-1} \equiv p + 1 \pmod{p^2}$, we can get at least one prime divisor of $(p^p - 1)/(p-1)$ which is not congruent to 1 modulo p^2. Denote such a prime divisor by q. This q is what we wanted. The proof is as follows. Assume that there exists an integer n such that $n^p \equiv p \pmod{q}$. Then we have $n^{p2} \equiv p^p \equiv 1 \pmod{q}$ by the definition of q. On the other hand, from Fermat's little theorem, $n^{p-1} \equiv 1 \pmod{q}$ because q is a prime. Since $p^2 \equiv \alpha \pmod{q-1}$, **we have** $(p^2, q-1) | p$, **which leads to** $n^p \equiv 1 \pmod{q}$. Hence we have $p \equiv 1 \pmod{q}$. However, this implies $1 + p + p^2 + \cdots + p^{p-1} \equiv p \pmod{q}$. From the definition of q, this leads to $p \equiv 0 \pmod{q}$, a contradiction.

47th INTERNATIONAL MATHEMATICAL OLYMPIAD SLOVENIA 2006

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Problems with Solutions

Solutions

Problem 1.

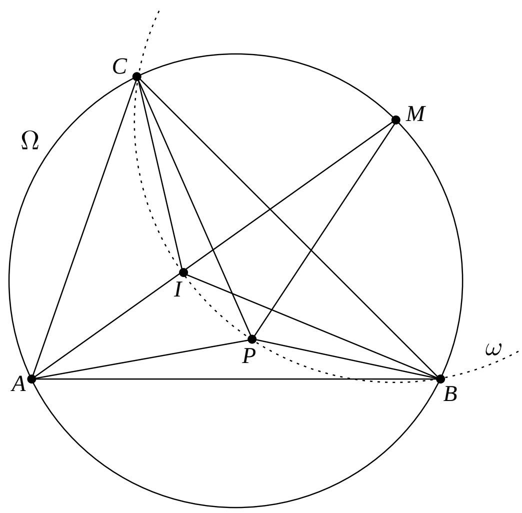
Let ABC be a triangle with incentre I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

Solution. Let $\angle A = \alpha$, $\angle B = \beta$, $\angle C = \gamma$. Since $\angle PBA + \angle PCA + \angle PBC + \angle PCB = \beta + \gamma$, the condition from the problem statement is equivalent to $\angle PBC + \angle PCB = (\beta + \gamma)/2$, i. e. $\angle BPC = 90^\circ + \alpha/2$.

On the other hand $\angle BIC = 180^\circ - (\beta + \gamma)/2 = 90^\circ + \alpha/2$. Hence $\angle BPC = \angle BIC$, and since P and I are on the same side of BC, the points B, C, I and P are concyclic. In other words, P lies on the circumcircle ω of triangle BCI.



Let Ω be the circumcircle of triangle ABC. It is a well-known fact that the centre of ω is the midpoint M of the arc BC of Ω. This is also the point where the angle bisector AI intersects Ω.

From triangle APM we have

$$AP + PM \geq AM = AI + IM = AI + PM.$$

Therefore $AP \geq AI$. Equality holds if and only if P lies on the line segment AI, which occurs if and only if $P = I$.

Problem 2.

Let P be a regular 2006-gon. A diagonal of P is called *good* if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called *good*.

Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

Solution 1. Call an isosceles triangle *good* if it has two odd sides. Suppose we are given a dissection as in the problem statement. A triangle in the dissection which is good and isosceles will be called *iso-good* for brevity.

Lemma. Let AB be one of dissecting diagonals and let $\mathcal{L}$ be the shorter part of the boundary of the 2006-gon with endpoints A, B. Suppose that $\mathcal{L}$ consists of n segments. Then the number of iso-good triangles with vertices on $\mathcal{L}$ does not exceed $n/2$.

Proof. This is obvious for $n = 2$. Take n with $2 < n \leq 1003$ and assume the claim to be true for every $\mathcal{L}$ of length less than n. Let now $\mathcal{L}$ (endpoints A, B) consist of n segments. Let PQ be the longest diagonal which is a side of an iso-good triangle PQS with all vertices on $\mathcal{L}$ (if there is no such triangle, there is nothing to prove). Every triangle whose vertices lie on $\mathcal{L}$ is obtuse or right-angled; thus S is the summit of PQS. We may assume that the five points A, P, S, Q, B lie on $\mathcal{L}$ in this order and partition $\mathcal{L}$ into four pieces $\mathcal{L}_{AP}$, $\mathcal{L}_{PS}$, $\mathcal{L}_{SQ}$, $\mathcal{L}_{QB}$ (the outer ones possibly reducing to a point).

By the definition of PQ, an iso-good triangle cannot have vertices on both $\mathcal{L}_{AP}$ and $\mathcal{L}_{QB}$. Therefore every iso-good triangle within $\mathcal{L}$ has all its vertices on just one of the four pieces. Applying to each of these pieces the induction hypothesis and adding the four inequalities we get that the number of iso-good triangles within $\mathcal{L}$ other than PQS does not exceed $n/2$. And since each of $\mathcal{L}_{PS}$, $\mathcal{L}_{SQ}$ consists of an odd number of sides, the inequalities for these two pieces are actually strict, leaving a $1/2 + 1/2$ in excess. Hence the triangle PSQ is also covered by the estimate $n/2$. This concludes the induction step and proves the lemma. □

The remaining part of the solution in fact repeats the argument from the above proof. Consider the longest dissecting diagonal XY. Let $\mathcal{L}_{XY}$ be the shorter of the two parts of the boundary with endpoints X, Y and let XYZ be the triangle in the dissection with vertex Z not on $\mathcal{L}_{XY}$. Notice that XYZ is acute or right-angled, otherwise one of the segments XZ, YZ would be longer than XY. Denoting by $\mathcal{L}_{XZ}$, $\mathcal{L}_{YZ}$ the two pieces defined by Z and applying the lemma to each of $\mathcal{L}_{XY}$, $\mathcal{L}_{XZ}$, $\mathcal{L}_{YZ}$ we infer that there are no more than 2006/2 iso-good triangles in all, unless XYZ is one of them. But in that case XZ and YZ are good diagonals and the corresponding inequalities are strict. This shows that also in this case the total number of iso-good triangles in the dissection, including XYZ, is not greater than 1003.

This bound can be achieved. For this to happen, it just suffices to select a vertex of the 2006-gon and draw a broken line joining every second vertex, starting from the selected one. Since 2006 is even, the line closes. This already gives us the required 1003 iso-good triangles. Then we can complete the triangulation in an arbitrary fashion.

Problem 3.

Determine the least real number M such that the inequality

$$\left| ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2) \right| \leq M\left(a^2 + b^2 + c^2\right)^2$$

holds for all real numbers a, b and c.

Solution. We first consider the cubic polynomial

$$P(t) = tb(t^2 - b^2) + bc(b^2 - c^2) + ct(c^2 - t^2).$$

It is easy to check that $P(b) = P(c) = P(-b - c) = 0$, and therefore

$$P(t) = (b - c)(t - b)(t - c)(t + b + c),$$

since the cubic coefficient is $b-c$. The left-hand side of the proposed inequality can therefore be written in the form

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)| = |P(a)| = |(b-c)(a-b)(a-c)(a+b+c)|.$$

The problem comes down to finding the smallest number M that satisfies the inequality

$$|(b-c)(a-b)(a-c)(a+b+c)| \le M\cdot(a^2+b^2+c^2)^2. \tag{1}$$

Note that this expression is symmetric, and we can therefore assume $a \le b \le c$ without loss of generality. With this assumption,

$$|(a-b)(b-c)| = (b-a)(c-b) \le \left(\frac{(b-a)+(c-b)}{2}\right)^2 = \frac{(c-a)^2}{4}, \tag{2}$$

with equality if and only if $b-a=c-b$, i.e. $2b=a+c$. Also

$$\left(\frac{(c-b)+(b-a)}{2}\right)^2 \le \frac{(c-b)^2+(b-a)^2}{2},$$

or equivalently,

$$3(c-a)^2 \le 2\cdot[(b-a)^2+(c-b)^2+(c-a)^2], \tag{3}$$

again with equality only for $2b=a+c$. From (2) and (3) we get

$$\begin{aligned}
&|(b-c)(a-b)(a-c)(a+b+c)|\\
\le\ & \frac{1}{4}\cdot|(c-a)^3(a+b+c)|\\
=\ & \frac{1}{4}\cdot\sqrt{(c-a)^6(a+b+c)^2}\\
\le\ & \frac{1}{4}\cdot\sqrt{\left(\frac{2\cdot[(b-a)^2+(c-b)^2+(c-a)^2]}{3}\right)^3\cdot(a+b+c)^2}\\
=\ & \frac{\sqrt{2}}{2}\cdot\left(\sqrt[4]{\left(\frac{(b-a)^2+(c-b)^2+(c-a)^2}{3}\right)^3\cdot(a+b+c)^2}\right)^2.
\end{aligned}$$

By the weighted AM-GM inequality this estimate continues as follows:

$$\begin{aligned}
&|(b-c)(a-b)(a-c)(a+b+c)|\\
\le\ & \frac{\sqrt{2}}{2}\cdot\left(\frac{(b-a)^2+(c-b)^2+(c-a)^2+(a+b+c)^2}{4}\right)^2\\
=\ & \frac{9\sqrt{2}}{32}\cdot(a^2+b^2+c^2)^2.
\end{aligned}$$

We see that the inequality (1) is satisfied for $M=\frac{9}{32}\sqrt{2}$, with equality if and only if $2b=a+c$ and

$$\frac{(b-a)^2+(c-b)^2+(c-a)^2}{3} = (a+b+c)^2.$$

Plugging $b=(a+c)/2$ into the last equation, we bring it to the equivalent form

$$2(c-a)^2 = 9(a+c)^2.$$

The conditions for equality can now be restated as

$$2b = a+c \qquad \text{and} \qquad (c-a)^2 = 18b^2.$$

Setting $b=1$ yields $a=1-\frac{3}{2}\sqrt{2}$ and $c=1+\frac{3}{2}\sqrt{2}$. We see that $M=\frac{9}{32}\sqrt{2}$ is indeed the smallest constant satisfying the inequality, with equality for any triple (a,b,c) proportional to $\left(1-\frac{3}{2}\sqrt{2},1,1+\frac{3}{2}\sqrt{2}\right)$, up to permutation.

Comment. With the notation $x = b - a$, $y = c - b$, $z = a - c$, $s = a + b + c$ and $r^2 = a^2 + b^2 + c^2$, the inequality (1) becomes just $|sxyz| \leq Mr^4$ (with suitable constraints on s and r). The original asymmetric inequality turns into a standard symmetric one; from this point on the solution can be completed in many ways. One can e.g. use the fact that, for fixed values of $\sum x$ and $\sum x^2$, the product xyz is a maximum/minimum only if some of x, y, z are equal, thus reducing one degree of freedom, etc. A specific attraction of the problem is that the maximum is attained at a point (a, b, c) with all coordinates distinct.

Problem 4.

Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

Solution. If (x, y) is a solution then obviously $x \geq 0$ and $(x, -y)$ is a solution too. For $x = 0$ we get the two solutions $(0, 2)$ and $(0, -2)$.

Now let (x, y) be a solution with $x > 0$; without loss of generality confine attention to $y > 0$. The equation rewritten as

$$2^x(1 + 2^{x+1}) = (y - 1)(y + 1)$$

shows that the factors $y - 1$ and $y + 1$ are even, exactly one of them divisible by 4. Hence $x \geq 3$ and one of these factors is divisible by 2^{x-1} but not by 2^x. So

$$y = 2^{x-1}m + \epsilon, \qquad m \ \text{ odd}, \qquad \epsilon = \pm 1. \tag{1}$$

Plugging this into the original equation we obtain

$$2^x\left(1 + 2^{x+1}\right) = \left(2^{x-1}m + \epsilon\right)^2 - 1 = 2^{2x-2}m^2 + 2^x m\epsilon,$$

or, equivalently

$$1 + 2^{x+1} = 2^{x-2}m^2 + m\epsilon.$$

Therefore

$$1 - \epsilon m = 2^{x-2}(m^2 - 8). \tag{2}$$

For $\epsilon = 1$ this yields $m^2 - 8 \leq 0$, i.e., $m = 1$, which fails to satisfy (2).

For $\epsilon = -1$ equation (2) gives us

$$1 + m = 2^{x-2}(m^2 - 8) \geq 2(m^2 - 8),$$

implying $2m^2 - m - 17 \leq 0$. Hence $m \leq 3$; on the other hand m cannot be 1 by (2). Because m is odd, we obtain $m = 3$, leading to $x = 4$. From (1) we get $y = 23$. These values indeed satisfy the given equation. Recall that then $y = -23$ is also good. Thus we have the complete list of solutions (x, y): $(0, 2)$, $(0, -2)$, $(4, 23)$, $(4, -23)$.

Problem 5.

Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

Solution. The claim is obvious if every integer fixed point of Q is a fixed point of P itself. For the sequel assume that this is not the case. Take any integer x_0 such that $Q(x_0) = x_0$, $P(x_0) \neq x_0$ and define inductively $x_{i+1} = P(x_i)$ for $i = 0, 1, 2, \ldots$; then $x_k = x_0$.

It is evident that

$$P(u) - P(v) \text{ is divisible by } u - v \text{ for distinct integers } u, v. \tag{1}$$

(Indeed, if $P(x) = \sum a_i x^i$ then each $a_i(u^i - v^i)$ is divisible by $u - v$.) Therefore each term in the chain of (nonzero) differences

$$x_0 - x_1, \quad x_1 - x_2, \quad \ldots, \quad x_{k-1} - x_k, \quad x_k - x_{k+1} \tag{2}$$

is a divisor of the next one; and since $x_k - x_{k+1} = x_0 - x_1$, all these differences have equal absolute values. For $x_m = \min(x_1, \ldots, x_k)$ this means that $x_{m-1} - x_m = -(x_m - x_{m+1})$. Thus $x_{m-1} = x_{m+1} (\neq x_m)$. It follows that consecutive differences in the sequence (2) have opposite signs. Consequently, $x_0, x_1, x_2, \ldots$ is an alternating sequence of two distinct values. In other words, every integer fixed point of Q is a fixed point of the polynomial $P(P(x))$. Our task is to prove that there are at most n such points.

Let a be one of them so that $b = P(a) \neq a$ (we have assumed that such an a exists); then $a = P(b)$. Take any other integer fixed point α of $P(P(x))$ and let $P(\alpha) = \beta$, so that $P(\beta) = \alpha$; the numbers α and β need not be distinct (α can be a fixed point of P), but each of α, β is different from each of a, b. Applying property (1) to the four pairs of integers (α, a), (β, b), (α, b), (β, a) we get that the numbers $\alpha - a$ and $\beta - b$ divide each other, and also $\alpha - b$ and $\beta - a$ divide each other. Consequently

$$\alpha - b = \pm(\beta - a), \qquad \alpha - a = \pm(\beta - b). \tag{3}$$

Suppose we have a plus in both instances: $\alpha - b = \beta - a$ and $\alpha - a = \beta - b$. Subtraction yields $a - b = b - a$, a contradiction, as $a \neq b$. Therefore at least one equality in (3) holds with a minus sign. For each of them this means that $\alpha + \beta = a + b$; equivalently $a + b - \alpha - P(\alpha) = 0$.

Denote $a + b$ by C. We have shown that every integer fixed point of Q other that a and b is a root of the polynomial $F(x) = C - x - P(x)$. This is of course true for a and b as well. And since P has degree $n > 1$, the polynomial F has the same degree, so it cannot have more than n roots. Hence the result.

Problem 6.

Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.

Solution 1.

Lemma. Every convex $(2n)$-gon, of area S, has a side and a vertex that jointly span a triangle of area not less than S/n.

Proof. By *main diagonals* of the $(2n)$-gon we shall mean those which partition the $(2n)$-gon into two polygons with equally many sides. For any side b of the $(2n)$-gon denote by Δ_b the triangle ABP where A, B are the endpoints of b and P is the intersection point of the main diagonals AA', BB'. We claim that the union of triangles Δ_b, taken over all sides, covers the whole polygon.

To show this, choose any side AB and consider the main diagonal AA' as a directed segment. Let X be any point in the polygon, not on any main diagonal. For definiteness, let X lie on the left side of the ray AA'. Consider the sequence of main diagonals AA', BB', CC', $\ldots$, where $A, B, C, \ldots$ are consecutive vertices, situated right to AA'.

The n-th item in this sequence is the diagonal $A'A$ (i.e. AA' reversed), having X on its right side. So there are two successive vertices K, L in the sequence $A, B, C, \ldots$ before A' such that X still lies to the left of KK' but to the right of LL'. And this means that X is in the triangle $\Delta_{\ell'}$, $\ell' = K'L'$. Analogous reasoning applies to points X on the right of AA' (points lying on main diagonals can be safely ignored). Thus indeed the triangles Δ_b jointly cover the whole polygon.

The sum of their areas is no less than S. So we can find two opposite sides, say $b = AB$ and $b' = A'B'$ (with AA', BB' main diagonals) such that $[\Delta_b] + [\Delta_{b'}] \geq S/n$, where $[\cdots]$ stands for the area of a region. Let AA', BB' intersect at P; assume without loss of generality that $PB \geq PB'$. Then

$$[ABA'] = [ABP] + [PBA'] \geq [ABP] + [PA'B'] = [\Delta_b] + [\Delta_{b'}] \geq S/n,$$

proving the lemma. $\square$

Now, let $\mathcal{P}$ be any convex polygon, of area S, with m sides $a_1, \ldots, a_m$. Let S_i be the area of the greatest triangle in $\mathcal{P}$ with side a_i. Suppose, contrary to the assertion, that

$$\sum_{i=1}^{m} \frac{S_i}{S} < 2.$$

Then there exist rational numbers $q_1, \ldots, q_m$ such that $\sum q_i = 2$ and $q_i > S_i/S$ for each i.

Let n be a common denominator of the m fractions $q_1, \ldots, q_m$. Write $q_i = k_i/n$; so $\sum k_i = 2n$. Partition each side a_i of $\mathcal{P}$ into k_i equal segments, creating a convex $(2n)$-gon of area S (with some angles of size 180°), to which we apply the lemma. Accordingly, this refined polygon has a side b and a vertex H spanning a triangle T of area $[T] \geq S/n$. If b is a piece of a side a_i of $\mathcal{P}$, then the triangle W with base a_i and summit H has area

$$[W] = k_i \cdot [T] \geq k_i \cdot S/n = q_i \cdot S > S_i,$$

in contradiction with the definition of S_i. This ends the proof.

Solution 2. As in the first solution, we allow again angles of size 180° at some vertices of the convex polygons considered.

To each convex n-gon $\mathcal{P} = A_1A_2 \ldots A_n$ we assign a centrally symmetric convex $(2n)$-gon $\mathcal{Q}$ with side vectors $\pm\overrightarrow{A_iA_{i+1}}$, $1 \leq i \leq n$. The construction is as follows. Attach the $2n$ vectors $\pm\overrightarrow{A_iA_{i+1}}$ at a common origin and label them $\overrightarrow{\mathbf{b}_1}, \overrightarrow{\mathbf{b}_2}, \ldots, \overrightarrow{\mathbf{b}_{2n}}$ in counterclockwise direction; the choice of the first vector $\overrightarrow{\mathbf{b}_1}$ is irrelevant. The order of labelling is well-defined if $\mathcal{P}$ has neither parallel sides nor angles equal to 180°. Otherwise several collinear vectors with the same direction are labelled consecutively $\overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}, \ldots, \overrightarrow{\mathbf{b}_{j+r}}$. One can assume that in such cases the respective opposite vectors occur in the order $-\overrightarrow{\mathbf{b}_j}, -\overrightarrow{\mathbf{b}_{j+1}}, \ldots, -\overrightarrow{\mathbf{b}_{j+r}}$, ensuring that $\overrightarrow{\mathbf{b}_{j+n}} = -\overrightarrow{\mathbf{b}_j}$ for $j = 1, \ldots, 2n$. Indices are taken cyclically here and in similar situations below.

Choose points $B_1, B_2, \ldots, B_{2n}$ satisfying $\overrightarrow{B_jB_{j+1}} = \overrightarrow{\mathbf{b}_j}$ for $j = 1, \ldots, 2n$. The polygonal line $\mathcal{Q} = B_1B_2 \ldots B_{2n}$ is closed, since $\sum_{j=1}^{2n} \overrightarrow{\mathbf{b}_j} = \overrightarrow{0}$. Moreover, $\mathcal{Q}$ is a convex $(2n)$-gon due to the arrangement of the vectors $\overrightarrow{\mathbf{b}_j}$, possibly with 180°-angles. The side vectors of $\mathcal{Q}$ are $\pm\overrightarrow{A_iA_{i+1}}$, $1 \leq i \leq n$. So in particular $\mathcal{Q}$ is centrally symmetric, because it contains as side vectors $\overrightarrow{A_iA_{i+1}}$ and $-\overrightarrow{A_iA_{i+1}}$ for each $i = 1, \ldots, n$. Note that B_jB_{j+1} and $B_{j+n}B_{j+n+1}$ are opposite sides of $\mathcal{Q}$, $1 \leq j \leq n$. We call $\mathcal{Q}$ the *associate* of $\mathcal{P}$.

Let S_i be the maximum area of a triangle with side A_iA_{i+1} in $\mathcal{P}$, $1 \leq i \leq n$. We prove that

$$[B_1B_2 \ldots B_{2n}] = 2\sum_{i=1}^{n} S_i \qquad (1)$$

and

$$[B_1B_2 \ldots B_{2n}] \geq 4\,[A_1A_2 \ldots A_n]\,. \qquad (2)$$

It is clear that (1) and (2) imply the conclusion of the original problem.

Lemma. For a side A_iA_{i+1} of $\mathcal{P}$, let h_i be the maximum distance from a point of $\mathcal{P}$ to line A_iA_{i+1}, $i = 1, \dots, n$. Denote by B_jB_{j+1} the side of $\mathcal{Q}$ such that $\overrightarrow{A_iA_{i+1}} = \overrightarrow{B_jB_{j+1}}$. Then the distance between B_jB_{j+1} and its opposite side in $\mathcal{Q}$ is equal to $2h_i$.

Proof. Choose a vertex A_k of $\mathcal{P}$ at distance h_i from line A_iA_{i+1}. Let $\mathbf{u}$ be the unit vector perpendicular to A_iA_{i+1} and pointing inside $\mathcal{P}$. Denoting by $\mathbf{x} \cdot \mathbf{y}$ the dot product of vectors $\mathbf{x}$ and $\mathbf{y}$, we have

$$h = \mathbf{u} \cdot \overrightarrow{A_iA_k} = \mathbf{u} \cdot (\overrightarrow{A_iA_{i+1}} + \dots + \overrightarrow{A_{k-1}A_k}) = \mathbf{u} \cdot (\overrightarrow{A_iA_{i-1}} + \dots + \overrightarrow{A_{k+1}A_k}).$$

In $\mathcal{Q}$, the distance H_i between the opposite sides B_jB_{j+1} and $B_{j+n}B_{j+n+1}$ is given by

$$H_i = \mathbf{u} \cdot (\overrightarrow{B_jB_{j+1}} + \dots + \overrightarrow{B_{j+n-1}B_{j+n}}) = \mathbf{u} \cdot (\overrightarrow{\mathbf{b}_j} + \overrightarrow{\mathbf{b}_{j+1}} + \dots + \overrightarrow{\mathbf{b}_{j+n-1}}).$$

The choice of vertex A_k implies that the n consecutive vectors $\overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}, \dots, \overrightarrow{\mathbf{b}_{j+n-1}}$ are precisely $\overrightarrow{A_iA_{i+1}}, \dots, \overrightarrow{A_{k-1}A_k}$ and $\overrightarrow{A_iA_{i-1}}, \dots, \overrightarrow{A_{k+1}A_k}$, taken in some order. This implies $H_i = 2h_i$. □

For a proof of (1), apply the lemma to each side of $\mathcal{P}$. If O the centre of $\mathcal{Q}$ then, using the notation of the lemma,

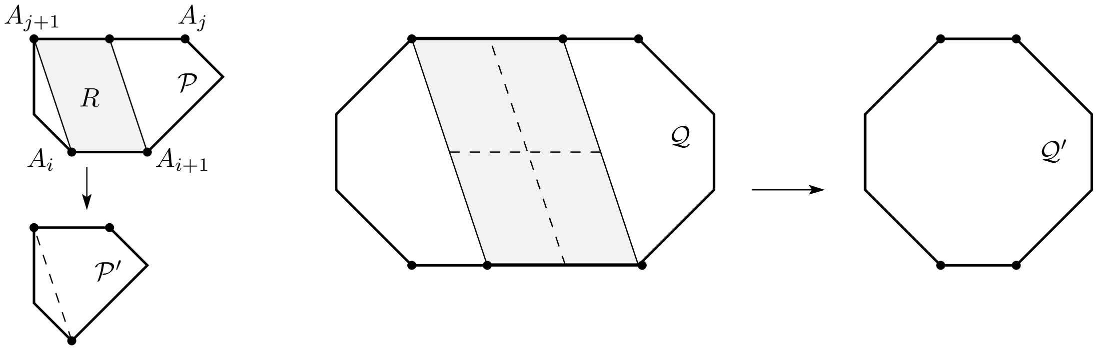
$$[B_jB_{j+1}O] = [B_{j+n}B_{j+n+1}O] = [A_iA_{i+1}A_k] = S_i\,.$$

Summation over all sides of $\mathcal{P}$ yields (1).

Set $d(\mathcal{P}) = [\mathcal{Q}] - 4[\mathcal{P}]$ for a convex polygon $\mathcal{P}$ with associate $\mathcal{Q}$. Inequality (2) means that $d(\mathcal{P}) \geq 0$ for each convex polygon $\mathcal{P}$. The last inequality will be proved by induction on the number ℓ of side directions of $\mathcal{P}$, i. e. the number of pairwise nonparallel lines each containing a side of $\mathcal{P}$.

We choose to start the induction with $\ell = 1$ as a base case, meaning that certain degenerate polygons are allowed. More exactly, we regard as *degenerate* convex polygons all closed polygonal lines of the form $X_1X_2 \dots X_kY_1Y_2 \dots Y_mX_1$, where $X_1, X_2, \dots, X_k$ are points in this order on a line segment X_1Y_1, and so are $Y_m, Y_{m-1}, \dots, Y_1$. The initial construction applies to degenerate polygons; their associates are also degenerate, and the value of d is zero. For the inductive step, consider a convex polygon $\mathcal{P}$ which determines ℓ side directions, assuming that $d(\mathcal{P}) \geq 0$ for polygons with smaller values of ℓ.

Suppose first that $\mathcal{P}$ *has a pair of parallel sides*, i. e. sides on distinct parallel lines. Let A_iA_{i+1} and A_jA_{j+1} be such a pair, and let $A_iA_{i+1} \leq A_jA_{j+1}$. Remove from $\mathcal{P}$ the parallelogram R determined by vectors $\overrightarrow{A_iA_{i+1}}$ and $\overrightarrow{A_iA_{j+1}}$. Two polygons are obtained in this way. Translating one of them by vector $\overrightarrow{A_iA_{i+1}}$ yields a new convex polygon $\mathcal{P}'$, of area $[\mathcal{P}] - [R]$ and with value of ℓ not exceeding the one of $\mathcal{P}$. The construction just described will be called operation $\mathbf{A}$.



The associate of $\mathcal{P}'$ is obtained from $\mathcal{Q}$ upon decreasing the lengths of two opposite sides by an amount of $2A_iA_{i+1}$. By the lemma, the distance between these opposite sides is twice the distance between A_iA_{i+1} and A_jA_{j+1}. Thus operation $\mathbf{A}$ decreases $[\mathcal{Q}]$ by the area of a parallelogram with base and respective altitude twice the ones of R, i. e. by $4[R]$. Hence $\mathbf{A}$ leaves the difference $d(\mathcal{P}) = [\mathcal{Q}] - 4[\mathcal{P}]$ unchanged.

Now, if $\mathcal{P}'$ also has a pair of parallel sides, apply operation $\mathbf{A}$ to it. Keep doing so with the subsequent polygons obtained for as long as possible. Now, $\mathbf{A}$ decreases the number p of pairs of parallel sides in $\mathcal{P}$. Hence its repeated applications gradually reduce p to 0, and further applications of $\mathbf{A}$ will be impossible after several steps. For clarity, let us denote by $\mathcal{P}$ again the polygon obtained at that stage.

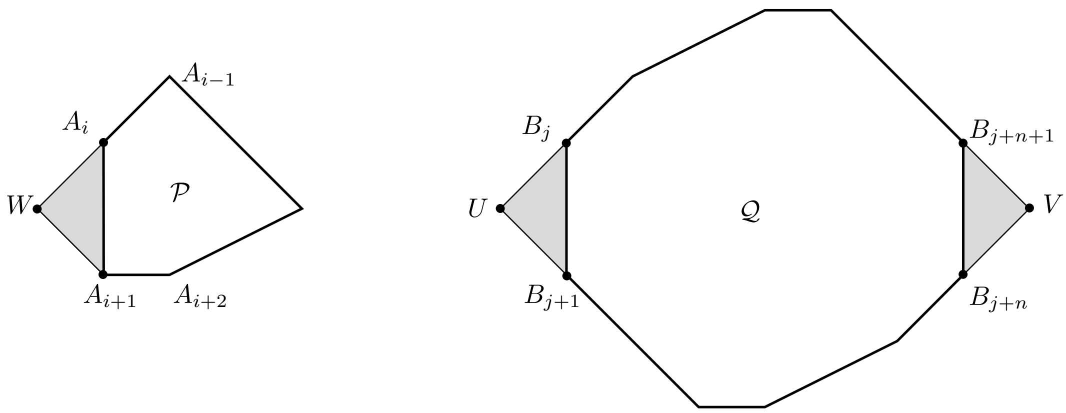
The inductive step is complete if $\mathcal{P}$ is degenerate. Otherwise $\ell > 1$ and $p = 0$, i. e. there are no parallel sides in $\mathcal{P}$. Observe that then $\ell \geq 3$. Indeed, $\ell = 2$ means that the vertices of $\mathcal{P}$ all lie on the boundary of a parallelogram, implying $p > 0$.

Furthermore, since $\mathcal{P}$ has no parallel sides, consecutive collinear vectors in the sequence $\left(\overrightarrow{\mathbf{b}_k}\right)$ (if any) correspond to consecutive 180°-angles in $\mathcal{P}$. Removing the vertices of such angles, we obtain a convex polygon with the same value of $d(\mathcal{P})$.

In summary, if operation $\mathbf{A}$ is impossible for a nondegenerate polygon $\mathcal{P}$, then $\ell \geq 3$. In addition, one may assume that $\mathcal{P}$ has no angles of size 180°.

The last two conditions then also hold for the associate $\mathcal{Q}$ of $\mathcal{P}$, and we perform the following construction. Since $\ell \geq 3$, there is a side B_jB_{j+1} of $\mathcal{Q}$ such that the sum of the angles at B_j and B_{j+1} is greater than 180°. (Such a side exists in each convex k-gon for $k > 4$.) Naturally, $B_{j+n}B_{j+n+1}$ is a side with the same property. Extend the pairs of sides $B_{j-1}B_j, B_{j+1}B_{j+2}$ and $B_{j+n-1}B_{j+n}, B_{j+n+1}B_{j+n+2}$ to meet at U and V, respectively. Let $\mathcal{Q}'$ be the centrally symmetric convex $2(n+1)$-gon obtained from $\mathcal{Q}$ by inserting U and V into the sequence $B_1, \ldots, B_{2n}$ as new vertices between B_j, B_{j+1} and B_{j+n}, B_{j+n+1}, respectively. Informally, we adjoin to $\mathcal{Q}$ the congruent triangles $B_jB_{j+1}U$ and $B_{j+n}B_{j+n+1}V$. Note that B_j, B_{j+1}, B_{j+n} and B_{j+n+1} are kept as vertices of $\mathcal{Q}'$, although B_jB_{j+1} and $B_{j+n}B_{j+n+1}$ are no longer its sides.

Let A_iA_{i+1} be the side of $\mathcal{P}$ such that $\overrightarrow{A_iA_{i+1}} = \overrightarrow{B_jB_{j+1}} = \overrightarrow{\mathbf{b}_j}$. Consider the point W such that triangle $A_iA_{i+1}W$ is congruent to triangle $B_jB_{j+1}U$ and exterior to $\mathcal{P}$. Insert W into the sequence $A_1, A_2, \ldots, A_n$ as a new vertex between A_i and A_{i+1} to obtain an $(n+1)$-gon $\mathcal{P}'$. We claim that $\mathcal{P}'$ is convex and its associate is $\mathcal{Q}'$.



Vectors $\overrightarrow{A_iW}$ and $\overrightarrow{\mathbf{b}_{j-1}}$ are collinear and have the same direction, as well as vectors $\overrightarrow{WA_{i+1}}$ and $\overrightarrow{\mathbf{b}_{j+1}}$. Since $\overrightarrow{\mathbf{b}_{j-1}}, \overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}$ are consecutive terms in the sequence $\left(\overrightarrow{\mathbf{b}_k}\right)$, the angle inequalities $\angle(\overrightarrow{\mathbf{b}_{j-1}}, \overrightarrow{\mathbf{b}_j}) \leq \angle(\overrightarrow{A_{i-1}A_i}, \overrightarrow{\mathbf{b}_j})$ and $\angle(\overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}) \leq \angle(\overrightarrow{\mathbf{b}_j}, \overrightarrow{A_{i+1}A_{i+2}})$ hold true. They show that $\mathcal{P}'$ is a convex polygon. To construct its associate, vectors $\pm\overrightarrow{A_iA_{i+1}} = \pm\overrightarrow{\mathbf{b}_j}$ must be deleted from the defining sequence $\left(\overrightarrow{\mathbf{b}_k}\right)$ of $\mathcal{Q}$, and the vectors $\pm\overrightarrow{A_iW}$, $\pm\overrightarrow{WA_{i+1}}$ must be inserted appropriately into it. The latter can be done as follows:

$$\ldots,\ \overrightarrow{\mathbf{b}_{j-1}},\ \overrightarrow{A_iW},\ \overrightarrow{WA_{i+1}},\ \overrightarrow{\mathbf{b}_{j+1}},\ \ldots,\ -\overrightarrow{\mathbf{b}_{j-1}},\ -\overrightarrow{A_iW},\ -\overrightarrow{WA_{i+1}},\ -\overrightarrow{\mathbf{b}_{j+1}},\ \ldots .$$

This updated sequence produces $\mathcal{Q}'$ as the associate of $\mathcal{P}'$.

It follows from the construction that $[\mathcal{P}'] = [\mathcal{P}] + [A_iA_{i+1}W]$ and $[\mathcal{Q}'] = [\mathcal{Q}] + 2[A_iA_{i+1}W]$. Therefore $d(\mathcal{P}') = d(\mathcal{P}) - 2[A_iA_{i+1}W] < d(\mathcal{P})$.

To finish the induction, it remains to notice that the value of ℓ for $\mathcal{P}'$ is less than the one for $\mathcal{P}$. This is because side A_iA_{i+1} was removed. The newly added sides A_iW and WA_{i+1} do not introduce new side directions. Each one of them is either parallel to a side of $\mathcal{P}$ or lies on the line determined by such a side. The proof is complete.

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Classical Inequalities

Ivan Matić

Contents

1 Introduction

This section will start with some basic facts and exercises. Frequent users of this discipline can just skim over the notation and take a look at formulas that talk about generalities in which the theorems will be shown.

The reason for starting with basic principles is the intention to show that the theory is simple enough to be completely derived on 20 pages without using any high-level mathematics. If you take a look at the first theorem and compare it with some scary inequality already mentioned in the table of contents, you will see how huge is the path that we will bridge in so few pages. And that will happen on a level accessible to a beginning high-school student. Well, maybe I exaggerated in the previous sentence, but the beginning high-school student should read the previous sentence again and forget about this one.

Theorem 1. *If x is a real number, then $x^2 \geq 0$. The equality holds if and only if $x = 0$.*

No proofs will be omitted in this text. Except for this one. We have to acknowledge that this is very important inequality, everything relies on it, ..., but the proof is so easy that it makes more sense wasting the space and time talking about its triviality than actually proving it. Do you know how to prove it? Hint: "A friend of my friend is my friend"; "An enemy of my enemy is my friend". It might be useful to notice that "An enemy of my friend is my enemy" and "A friend of my enemy is my enemy", but the last two facts are not that useful for proving theorem 1.

I should also write about the difference between "$\geq$" and "$>$"; that something weird happens when both sides of an inequality are multiplied by a negative number, but I can't imagine myself doing that. People would hate me for real.

Theorem 2. *If $a, b \in \mathbb{R}$ then:*

$$a^2 + b^2 \geq 2ab. \tag{1}$$

The equality holds if and only if $a = b$.

Proof. After subtracting $2ab$ from both sides the inequality becomes equivalent to $(a-b)^2 \geq 0$, which is true according to theorem 1. □

Problem 1. *Prove the inequality $a^2 + b^2 + c^2 \geq ab + bc + ca$, if a, b, c are real numbers.*

Solution. If we add the inequalities $a^2 + b^2 \geq 2ab$, $b^2 + c^2 \geq 2bc$, and $c^2 + a^2 \geq 2ca$ we get $2a^2 + 2b^2 + 2c^2 \geq 2ab + 2bc + 2ca$, which is equivalent to what we are asked to prove. △

Problem 2. *Find all real numbers a, b, c, and d such that*

$$a^2 + b^2 + c^2 + d^2 = a(b + c + d).$$

Solution. Recall that $x^2 + y^2 \geq 2xy$, where the equality holds if and only if $x = y$. Applying this inequality to the pairs of numbers $(a/2, b)$, $(a/2, c)$, and $(a/2, d)$ yields:

$$\frac{a^2}{4} + b^2 \geq ab, \quad \frac{a^2}{4} + c^2 \geq ac, \quad \frac{a^2}{4} + d^2 \geq ad.$$

Note also that $a^2/4 > 0$. Adding these four inequalities gives us $a^2 + b^2 + c^2 + d^2 \geq a(b + c + d)$. Equality can hold only if all the inequalities were equalities, i.e. $a^2 = 0$, $a/2 = b$, $a/2 = c$, $a/2 = d$. Hence $a = b = c = d = 0$ is the only solution of the given equation. △

Problem 3. *If a, b, c are positive real numbers that satisfy $a^2 + b^2 + c^2 = 1$, find the minimal value of*

$$S = \frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2}.$$

Solution. If we apply the inequality $x^2 + y^2 \geq 2xy$ to the numbers $x = \frac{ab}{c}$ and $y = \frac{bc}{a}$ we get

$$\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} \geq 2b^2. \tag{2}$$

Similarly we get

$$\frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2} \geq 2c^2, \text{ and} \tag{3}$$

$$\frac{c^2a^2}{b^2} + \frac{a^2b^2}{c^2} \geq 2a^2. \tag{4}$$

Summing up (2), (3), and (4) gives $2\left(\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2}\right) \geq 2(a^2 + b^2 + c^2) = 2$, hence $S \geq 1$. The equality holds if and only if $\frac{ab}{c} = \frac{bc}{a} = \frac{ca}{b}$, i.e. $a = b = c = \frac{1}{\sqrt{3}}$. △

Problem 4. *If x and y are two positive numbers less than* 1*, prove that*

$$\frac{1}{1-x^2}+\frac{1}{1-y^2}\geq\frac{2}{1-xy}.$$

Solution. Using the inequality $a+b\geq 2\sqrt{ab}$ we get $\frac{1}{1-x^2}+\frac{1}{1-y^2}\geq\frac{2}{\sqrt{(1-x^2)(1-y^2)}}$. Now we notice that $(1-x^2)(1-y^2)=1+x^2y^2-x^2-y^2\leq 1+x^2y^2-2xy=(1-xy)^2$ which implies $\frac{2}{\sqrt{(1-x^2)(1-y^2)}}\geq\frac{2}{1-xy}$ and this completes the proof. △

Since the main focus of this text is to present some more advanced material, the remaining problems will be harder then the ones already solved. For those who want more of the introductory-type problems, there is a real hope that this website will soon get some text of that sort. However, nobody should give up from reading the rest, things are getting very interesting.

Let us return to the inequality (1) and study some of its generalizations. For $a,b\geq 0$, the consequence $\frac{a+b}{2}\geq\sqrt{ab}$ of (1) is called the Arithmetic-Geometric mean inequality. Its left-hand side is called the arithmetic mean of the numbers *a* and *b*, and its right-hand side is called the geometric mean of *a* and *b*. This inequality has its analogue:

$$\frac{a+b+c}{3}\geq\sqrt[3]{abc},\ a,b,c\geq 0.$$

More generally, for a sequence $x_1,\ldots,x_n$ of positive real numbers, the Arithmetic-Geometric mean inequality holds:

$$\frac{x_1+x_2+\cdots+x_n}{n}\geq\sqrt[n]{x_1\cdot x_2\cdots x_n}. \tag{5}$$

These two inequalities are highly non-trivial, and there are variety of proofs to them. We did (5) for $n=2$. If you try to prove it for $n=3$, you would see the real trouble. What a person tortured with the case $n=3$ would never suspect is that $n=4$ is much easier to handle. It has to do something with 4 being equal $2\cdot 2$ and $3\neq 2\cdot 2$. I believe you are not satisfied by the previous explanation but you have to accept that the case $n=3$ comes after the case $n=4$. The induction argument follows these lines, but (un)fortunately we won't do it here because that method doesn't allow generalizations that we need.

Besides (5) we have the inequality between quadratic and arithmetic mean, namely

$$\sqrt{\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}}\geq\frac{x_1+x_2+\cdots+x_n}{n}. \tag{6}$$

The case of equality in (5) and (6) occurs if and only if all the numbers $x_1,\ldots,x_n$ are equal.

Arithmetic, geometric, and quadratic means are not the only means that we will consider. There are infinitely many of them, and there are infinitely many inequalities that generalize (5) and (6). The beautiful thing is that we will consider all of them at once. For appropriately defined means, a very general inequality will hold, and the above two inequalities will ended up just being consequences.

Definition 1. *Given a sequence* $x_1,x_2,\ldots,x_n$ *of positive real numbers, the mean of order r, denoted by* $M_r(x)$ *is defined as*

$$M_r(x)=\left(\frac{x_1^r+x_2^r+\cdots+x_n^r}{n}\right)^{\frac{1}{r}}. \tag{7}$$

Example 1. *$M_1(x_1,\dots,x_n)$ is the arithmetic mean, while $M_2(x_1,\dots,x_n)$ is the geometric mean of the numbers $x_1,\dots,x_n$.*

M_0 can't be defined using the expression (7) but we will show later that as r approaches 0, M_r will approach the geometric mean. The famous mean inequality can be now stated as

$$M_r(x_1,\dots,x_n) \leq M_s(x_1,\dots,x_n), \quad \text{for } 0 \leq r \leq s.$$

However we will treat this in slightly greater generality.

Definition 2. *Let $m = (m_1,\dots,m_n)$ be a fixed sequence of non-negative real numbers such that $m_1 + m_2 + \cdots + m_n = 1$. Then the weighted mean of order r of the sequence of positive reals $x = (x_1,\dots,x_n)$ is defined as:*

$$M_r^m(x) = \left(x_1^r m_1 + x_2^r m_2 + \cdots + x_n^r m_n\right)^{\frac{1}{r}}. \tag{8}$$

Remark. Sequence m is sometimes called a sequence of masses, but more often it is called a measure, and $M_r^m(x)$ is the L^r norm with repsect to the Lebesgue integral defined by m. I didn't want to scare anybody. I just wanted to emphasize that this hard-core math and not something coming from physics.

We will prove later that as r tends to 0, the weighted mean $M_r^m(x)$ will tend to the weighted geometric mean of the sequence x defined by $G^m(x) = x_1^{m_1} \cdot x_2^{m_2} \cdots x_n^{m_n}$.

Example 2. *If $m_1 = m_2 = \cdots = \frac{1}{n}$ then $M_r^m(x) = M_r(x)$ where $M_r(x)$ is previously defined by the equation (7).*

Theorem 3 (General Mean Inequality)**.** *If $x = (x_1,\dots,x_n)$ is a sequence of positive real numbers and $m = (m_1,\dots,m_n)$ another sequence of positive real numbers satisfying $m_1 + \cdots + m_n = 1$, then for $0 \leq r \leq s$ we have $M_r^m(x) \leq M_s^m(x)$.*

The proof will follow from the Hölders inequality.

2 Convex Funtions

To prove some of the fundamental results we will need to use convexity of certain functions. Proofs of the theorems of Young, Minkowski, and Hölder will require us to use very basic facts – you should be fine if you just read the definition 3 and example 3. However, the section on Karamata's inequality will require some deeper knowledge which you can find here.

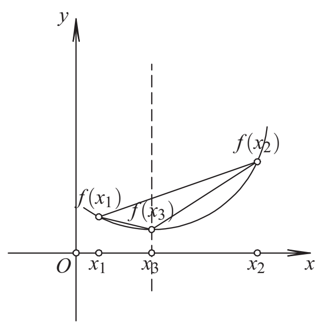
Definition 3. *The function $f : [a,b] \to \mathbb{R}$ is convex if for any $x_1, x_2 \in [a,b]$ and any $\lambda \in (0,1)$ the following inequality holds:*

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2). \tag{9}$$

Function is called concave if $-f$ is convex. If the inequality in (9) is strict then the function is called strictly convex.

Now we will give a geometrical interpretation of convexity. Take any $x_3 \in (x_1, x_2)$. There is $\lambda \in (0,1)$ such that $x_2 = \lambda x_1 + (1-\lambda)x_3$. Let's paint in green the line passing through x_3 and parallel to the y axis. Let's paint in red the chord connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Assume that the green line and the red chord intersect at the yellow point. The y coordinate (also called the height) of the yellow point is:

$$\lambda f(x_1) + (1-\lambda) f(x_2).$$



The inequality (9) means exactly that the the green line will intersect the graph of a function below the red chord. If f is strictly convex then the equality can hold in (9) if and only if $x_1 = x_2$.

Example 3. *The following functions are convex:* e^x, x^p *(for* $p \geq 1$, $x > 0$*),* $\frac{1}{x}$ *(*$x \neq 0$*), while the functions* $\log x$ *(*$x > 0$*),* $\sin x$ *(*$0 \leq x \leq \pi$*) ,* $\cos x$ *(*$-\pi/2 \leq x \leq \pi/2$*) are concave.*

All functions mentioned in the previous example are elementary functions, and proving the convexity/concavity for them would require us to go to the very basics of their foundation, and we will not do that. In many of the examples and problems respective functions are slight modifications of elementary functions. Their convexity (or concavity) is something we don't have to verify. However, we will develop some criteria for verifying the convexity of more complex combinations of functions.

Let us take another look at our picture above and compare the slopes of the three drawn lines. The line connecting $(x_1, f(x_1))$ with $(x_3, f(x_3))$ has the smallest slope, while the line connecting $(x_3, f(x_3))$ with $(x_2, f(x_2))$ has the largest slope. In the following theorem we will state and prove that the convex function has always an "increasing slope".

Theorem 4. *Let* $f : [a,b] \to \mathbb{R}$ *be a convex function and* $a \leq x_1 < x_3 < x_2 \leq b$. *Then*

$$\frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_2) - f(x_3)}{x_2 - x_3}. \tag{10}$$

Proof. We can write $x_3 = \lambda x_1 + (1-\lambda)x_2$ for some $\lambda \in (0,1)$. More precisely $\lambda = \frac{x_2 - x_3}{x_2 - x_1}$, and $1 - \lambda = \frac{x_3 - x_1}{x_2 - x_1}$. From (9) we get

$$f(x_3) \leq \frac{x_2 - x_3}{x_2 - x_1} f(x_1) + \frac{x_3 - x_1}{x_2 - x_1} f(x_2).$$

Subtracting $f(x_1)$ from both sides of the last inequality yields $f(x_3) - f(x_1) = -\frac{x_3 - x_1}{x_2 - x_1} f(x_1) + \frac{x_3 - x_1}{x_2 - x_1} f(x_2)$ giving immediately the first inequality of (10). The second inequality of (10) is obtained in an analogous way. □

The rest of this chapter is using some of the properties of limits, continuity and differentiability. If you are not familiar with basic calculus, you may skip that part, and you will be able to understand most of what follows. The theorem 6 is the tool for verifying the convexity for differentiable functions that we mentioned before. The theorem 5 will be used it in the proof of Karamata's inequality.

Theorem 5. *If* $f : (a,b) \to \mathbb{R}$ *is a convex function, then* f *is continuous and at every point* $x \in (a,b)$ *it has both left and right derivative* $f'_-(x)$ *and* $f'_+(x)$. *Both* f'_- *and* f'_+ *are increasing functions on* (a,b) *and* $f'_-(x) \leq f'_+(x)$.

Solution. The theorem 10 implies that for fixed x the function $\varphi(t)=\frac{f(t)-f(x)}{t-x}$, $t\neq x$ is an increasing function bounded both by below and above. More precisely, if t_0 and t_1 are any two numbers from (a,b) such that $t_0<x<t_1$ we have:

$$\frac{f(x)-f(t_0)}{x-t_0}\leq\varphi(t)\leq\frac{f(t_1)-f(x)}{t_1-x}.$$

This specially means that there are $\lim_{t\to x-}\varphi(t)$ and $\lim_{t\to x+}\varphi(t)$. The first one is precisely the left, and the second one – the right derivative of φ at x. Since the existence of both left and right derivatives implies the continuity, the statement is proved. □

Theorem 6. *If $f:(a,b)\to\mathbb{R}$ is a twice differentiable function. Then f is convex on (a,b) if and only if $f''(x)\geq 0$ for every $x\in(a,b)$. Moreover, if $f''(x)>0$ then f is strictly convex.*

Proof. This theorem is the immediate consequence of the previous one. □

3 Inequalities of Minkowski and Hölder

Inequalities presented here are sometimes called weighted inequalities of Minkowski, Hölder, and Cauchy-Schwartz. The standard inequalities are easily obtained by placing $m_i=1$ whenever some m appears in the text below. Assuming that the sum $m_1+\cdots+m_n=1$ one easily get the generalized (weighted) mean inequalities, and additional assumption $m_i=1/n$ gives the standard mean inequalities.

Lemma 1. *If $x,y>0$, $p>1$ and $\alpha\in(0,1)$ are real numbers, then*

$$(x+y)^p\leq\alpha^{1-p}x^p+(1-\alpha)^{1-p}y^p. \qquad (11)$$

The equality holds if and only if $\frac{x}{\alpha}=\frac{y}{1-\alpha}$.

Proof. For $p>1$, the function $\varphi(x)=x^p$ is strictly convex hence $(\alpha a+(1-\alpha)b)^p\leq\alpha a^p+(1-\alpha)b^p$. The equality holds if and only if $a=b$. Setting $x=\alpha a$ and $y=(1-\alpha)b$ we get (11) immediately. □

Lemma 2. *If $x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n$ and $m_1,m_2,\dots,m_n$ are three sequences of positive real numbers and $p>1$, $\alpha\in(0,1)$, then*

$$\sum_{i=1}^n(x_i+y_i)^pm_i\leq\alpha^{1-p}\sum_{i=1}^nx_i^pm_i+(1-\alpha)^{1-p}\sum_{i=1}^ny_i^pm_i. \qquad (12)$$

The equality holds if and only if $\frac{x_i}{y_i}=\frac{\alpha}{1-\alpha}$ for every i, $1\leq i\leq n$.

Proof. From (11) we get $(x_i+y_i)^p\leq\alpha^{1-p}x_i^p+(1-\alpha)^{1-p}y_i^p$. Multiplying by m_i and adding as $1\leq i\leq n$ we get (12). The equality holds if and only if $\frac{x_i}{y_i}=\frac{\alpha}{1-\alpha}$. □

Theorem 7 (Minkowski)**.** *If x_1, x_2, …, x_n, y_1, y_2, …, y_n, and m_1, m_2, …, m_n are three sequences of positive real numbers and $p>1$, then*

$$\left(\sum_{i=1}^n(x_i+y_i)^pm_i\right)^{1/p}\leq\left(\sum_{i=1}^nx_i^pm_i\right)^{1/p}+\left(\sum_{i=1}^ny_i^pm_i\right)^{1/p}. \qquad (13)$$

The equality holds if and only if the sequences (x_i) and (y_i) are proportional, i.e. if and only if there is a constant λ such that $x_i=\lambda y_i$ for $1\leq i\leq n$.

Proof. For any $\alpha \in (0,1)$ we have inequality (12). Let us write

$$A = \left(\sum_{i=1}^{n} x_i^p m_i\right)^{1/p}, \quad B = \left(\sum_{i=1}^{n} y_i^p m_i\right)^{1/p}.$$

In new terminology (12) reads as

$$\sum_{i=1}^{n} (x_i + y_i)^p m_i \le \alpha^{1-p} A^p + (1-\alpha)^{1-p} B^p. \tag{14}$$

If we choose α such that $\frac{A}{\alpha} = \frac{B}{1-\alpha}$, then (11) implies $\alpha^{1-p}A^p + (1-\alpha)^{1-p}B^p = (A+B)^p$ and (14) now becomes

$$\sum_{i=1}^{n} (x_i + y_i)^p m_i = \left[\left(\sum_{i=1}^{n} x_i^p m_i\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^p m_i\right)^{1/p}\right]^p$$

which is equivalent to (13). □

Problem 5 (SL70). *If $u_1, \dots, u_n, v_1, \dots, v_n$ are real numbers, prove that*

$$1 + \sum_{i=1}^{n} (u_i + v_i)^2 \le \frac{4}{3}\left(1 + \sum_{i=1}^{n} u_i^2\right)\left(1 + \sum_{i=1}^{n} v_i^2\right).$$

When does equality hold?

Solution. Let us set $a = \sqrt{\sum_{i=1}^{n} u_i^2}$ and $b = \sqrt{\sum_{i=1}^{n} v_i^2}$. By Minkowski's inequality (for $p = 2$) we have $\sum_{i=1}^{n}(u_i + v_i)^2 \le (a+b)^2$. Hence the LHS of the desired inequality is not greater than $1 + (a+b)^2$, while the RHS is equal to $4(1+a^2)(1+b^2)/3$. Now it is sufficient to prove that

$$3 + 3(a+b)^2 \le 4(1+a^2)(1+b^2).$$

The last inequality can be reduced to the trivial $0 \le (a-b)^2 + (2ab-1)^2$. The equality in the initial inequality holds if and only if $u_i/v_i = c$ for some $c \in \mathbb{R}$ and $a = b = 1/\sqrt{2}$. △

Theorem 8 (Young). *If $a, b > 0$ and $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}. \tag{15}$$

Equality holds if and only if $a^p = b^q$.

Proof. Since $\varphi(x) = e^x$ is a convex function we have that $e^{\frac{1}{p}x + \frac{1}{q}y} \le \frac{1}{p}e^x + \frac{1}{q}e^y$. The equality holds if and only if $x = y$, and the inequality (15) is immediately obtained by placing $a = e^{x/p}$ and $b = e^{y/q}$. The equality holds if and only if $a^p = b^q$. □

Lemma 3. *If $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, m_1, m_2, \dots, m_n$ are three sequences of positive real numbers and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and $\alpha > 0$, then*

$$\sum_{i=1}^{n} x_i y_i m_i \le \frac{1}{p} \cdot \alpha^p \cdot \sum_{i=1}^{n} x_i^p m_i + \frac{1}{q} \cdot \frac{1}{\alpha^q} \cdot \sum_{i=1}^{n} y_i^q m_i. \tag{16}$$

The equality holds if and only if $\frac{\alpha^p x_i^p}{p} = \frac{y_i^q}{q\alpha^q}$ for $1 \le i \le n$.

Proof. From (15) we immediately get $x_iy_i = (\alpha x_i)\frac{y_i}{\alpha} \leq \frac{1}{p}\cdot\alpha^p x_i^p + \frac{1}{q}\cdot\frac{1}{\alpha^q}y_i^q$. Multiplying by m_i and adding as $i = 1,2,\dots,n$ we get (16). The inequality holds if and only if $\frac{\alpha^p x_i^p}{p} = \frac{y_i^q}{q\alpha^q}$ for $1 \leq i \leq n$. □

Theorem 9 (Hölder). *If $x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n,m_1,m_2,\dots,m_n$ are three sequences of positive real numbers and $p,q > 1$ such that $\frac{1}{p}+\frac{1}{q} = 1$, then*

$$\sum_{i=1}^n x_iy_im_i \leq \left(\sum_{i=1}^n x_i^p m_i\right)^{1/p}\cdot\left(\sum_{i=1}^n y_i^q m_i\right)^{1/q}. \quad (17)$$

The equality holds if and only if the sequences (x_i^p) and (y_i^q) are proportional.

Proof. The idea is very similar to the one used in the proof of Minkowski's inequality. The inequality (16) holds for any positive constant α. Let

$$A = \left(\alpha^p\sum_{i=1}^n x_i^p m_i\right)^{1/p},\quad B = \left(\frac{1}{\alpha^q}\sum_{i=1}^n y_i^q m_i\right)^{1/q}.$$

By Young's inequality we have that $\frac{1}{p}A^p + \frac{1}{q}B^q = AB$ if $A^p = B^q$. Equivalently $\alpha^p\sum_{i=1}^n x_i^p m_i = \frac{1}{\alpha^q}\sum_{i=1}^n y_i^q m_i$. Choosing such an α we get

$$\sum_{i=1}^n x_iy_im_i \leq \frac{1}{p}A^p + \frac{1}{q}B^q = AB = \left(\sum_{i=1}^n x_i^p m_i\right)^{1/p}\cdot\left(\sum_{i=1}^n y_i^q m_i\right)^{1/q}. \;\square$$

Problem 6. *If $a_1,\dots,a_n$ and $m_1,\dots,m_n$ are two sequences of positive numbers such that $a_1m_1+\cdots+a_nm_n = \alpha$ and $a_1^2m_1+\cdots+a_n^2m_n = \beta^2$, prove that $\sqrt{a_1}m_1+\cdots+\sqrt{a_n}m_n \geq \frac{\alpha^{3/2}}{\beta}$.*

Solution. We will apply Hölder's inequality on $x_i = a_i^{1/3}$, $y_i = a_i^{2/3}$, $p = \frac{3}{2}$, $q = 3$:

$$\alpha = \sum_{i=1}^n a_im_i \leq \left(\sum_{i=1}^n a_i^{1/2}m_i\right)^{2/3}\cdot\left(\sum_{i=1}^n a_i^2 m_i\right)^{1/3} = \left(\sum_{i=1}^n \sqrt{a_i}m_i\right)^{2/3}\cdot\beta^{2/3}.$$

Hence $\sum_{i=1}^n \sqrt{a_i}m_i \geq \frac{\alpha^{3/2}}{\beta}$. △

Proof of the theorem 3. $M_r^m = \left(\sum_{i=1}^n x_i^r\cdot m_i\right)^{1/r}$. We will use the Hölders inequality for $y_i = 1$, $p = \frac{s}{r}$, and $q = \frac{p}{1-p}$. Then we get

$$M_r^m \leq \left(\sum_{i=1}^n x_i^{rp}\cdot m_i\right)^{\frac{1}{pr}}\cdot\left(\sum_{i=1}^n 1^q\cdot m_i\right)^{p/(1-p)} = M_s.\;\square$$

Problem 7. *(SL98) Let x, y, and z be positive real numbers such that $xyz = 1$. Prove that*

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

Solution. The given inequality is equivalent to

$$x^3(x+1)+y^3(y+1)+z^3(z+1) \geq \frac{3}{4}(1+x+y+z+xy+yz+zx+xyz).$$

The left-hand side can be written as $x^4+y^4+z^4+x^3+y^3+z^3 = 3M_4^4+3M_3^3$. Using $xy+yz+zx \leq x^2+y^2+z^2 = 3M_2^2$ we see that the right-hand side is less than or equal to $\frac{3}{4}(2+3M_1+3M_2^2)$. Since $M_1 \geq 3\sqrt[3]{xyz} = 1$, we can further say that the right-hand side of the required inequality is less than or equal to $\frac{3}{4}(5M_1+3M_2^2)$. Since $M_4 \geq M_3$, and $M_1 \leq M_2 \leq M_3$, the following inequality would imply the required statement:

$$3M_3^4+3M_3^3 \geq \frac{3}{4}(5M_3+3M_3^2).$$

However the last inequality is equivalent to $(M_3-1)(4M_3^2+8M_3+5) \geq 0$ which is true because $M_3 \geq 1$. The equality holds if and only if $x=y=z=1$. △

Theorem 10 (Weighted Cauchy-Schwartz)**.** *If x_i, y_i are real numbers, and m_i positive real numbers, then*

$$\sum_{i=1}^{n} x_i y_i m_i \quad \leq \quad \sqrt{\sum_{i=1}^{n} x_i^2 m_i} \cdot \sqrt{\sum_{i=1}^{n} y_i^2 m_i}. \tag{18}$$

Proof. After noticing that $\sum_{i=1}^{n} x_i y_i m_i \leq \sum_{i=1}^{n} |x_i| \cdot |y_i| m_i$, the rest is just a special case ($p=q=2$) of the Hölder's inequality. □

Problem 8. *If a, b, and c are positive numbers, prove that*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{(a+b+c)^2}{ab+bc+ca}.$$

Solution. We will apply the Cauchy-Schwartz inequality with $x_1 = \sqrt{\frac{a}{b}}$, $x_2 = \sqrt{\frac{b}{c}}$, $x_3 = \sqrt{\frac{c}{a}}$, $y_1 = \sqrt{ab}$, $y_2 = \sqrt{bc}$, and $y_3 = \sqrt{ca}$. Then

$$\begin{aligned} a+b+c &= x_1y_1+x_2y_2+x_3y_3 \leq \sqrt{x_1^2+x_2^2+x_3^2} \cdot \sqrt{y_1^2+y_2^2+y_3^2} \\ &= \sqrt{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}} \cdot \sqrt{ab+bc+ca}. \end{aligned}$$

Theorem 11. *If $a_1,\dots,a_n$ are positive real numbers, then*

$$\lim_{r\to 0} M_r(a_1,\dots,a_n) = a_1^{m_1} \cdot a_2^{m_2} \cdots a_n^{m_n}.$$

Proof. This theorem is given here for completeness. It states that as $r \to 0$ the mean of order r approaches the geometric mean of the sequence. Its proof involves some elementary calculus, and the reader can omit the proof.

$$M_r(a_1,\dots,a_n) = e^{\frac{1}{r}\log(a_1^r m_1+\cdots+a_n^r m_n)}.$$

Using the L'Hospitale's theorem we get

$$\begin{aligned} \lim_{r\to 0} \frac{1}{r} \log(a_1^r m_1+\cdots+a_n^r m_n) &= \lim_{r\to 0} \frac{m_1 a_1^r \log a_1+\cdots+m_n a_n^r \log a_n}{a_1^r m_1+\cdots+a_n^r m_n} \\ &= m_1 \log a_1+\cdots+m_n \log a_n \\ &= \log\left(a_1^{m_1}\cdots a_n^{m_n}\right). \end{aligned}$$

The result immediately follows. □

4 Inequalities of Schur and Muirhead

Definition 4. *Let $\sum ! F(a_1,\dots,a_n)$ be the sum of $n!$ summands which are obtained from the function $F(a_1,\dots,a_n)$ making all permutations of the array (a).*

We will consider the special cases of the functio F, i.e. when $F(a_1,\dots,a_n)=a_1^{\alpha_1}\cdot\dots\cdot a_n^{\alpha_n}$, $\alpha_i\geq 0$.

If (α) is an array of exponents and $F(a_1,\dots,a_n)=a_1^{\alpha_1}\cdot\dots\cdot a_n^{\alpha_n}$ we will use $T[\alpha_1,\dots,\alpha_n]$ instead of $\sum ! F(a_1,\dots,a_n)$, if it is clear what is the sequence (a).

Example 4. *$T[1,0,\dots,0]=(n-1)!\cdot(a_1+a_2+\dots+a_n)$, and $T[\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n}]=n!\cdot\sqrt[n]{a_1\cdot\dots\cdot a_n}$. The AM-GM inequality is now expressed as:*

$$T[1,0,\dots,0]\geq T\left[\frac{1}{n},\dots,\frac{1}{n}\right].$$

Theorem 12 (Schur). *For $\alpha\in\mathbb{R}$ and $\beta>0$ the following inequality holds:*

$$T[\alpha+2\beta,0,0]+T[\alpha,\beta,\beta]\geq 2T[\alpha+\beta,\beta,0]. \tag{19}$$

Proof. Let (x,y,z) be the sequence of positive reals for which we are proving (19). Using some elementary algebra we get

$$\begin{aligned}&\frac{1}{2}T[\alpha+2\beta,0,0]+\frac{1}{2}T[\alpha,\beta,\beta]-T[\alpha+\beta,\beta,0]\\ =\ & x^{\alpha}(x^{\beta}-y^{\beta})(x^{\beta}-z^{\beta})+y^{\alpha}(y^{\beta}-x^{\beta})(y^{\beta}-z^{\beta})+z^{\alpha}(z^{\beta}-x^{\beta})(z^{\beta}-y^{\beta}).\end{aligned}$$

Without loss of generality we may assume that $x\geq y\geq z$. Then in the last expression only the second summand may be negative. If $\alpha\geq 0$ then the sum of the first two summands is ≥ 0 because $x^{\alpha}(x^{\beta}-y^{\beta})(x^{\beta}-z^{\beta})\geq x^{\alpha}(x^{\beta}-y^{\beta})(y^{\beta}-z^{\beta})\geq y^{\alpha}(x^{\beta}-y^{\beta})(y^{\beta}-z^{\beta})=-y^{\alpha}(x^{\beta}-y^{\beta})(y^{\beta}-z^{\beta})$. Similarly for $\alpha<0$ the sum of the last two terms is ≥ 0. □

Example 5. *If we set $\alpha=\beta=1$, we get*

$$x^3+y^3+z^3+3xyz\geq x^2y+xy^2+y^2z+yz^2+z^2x+zx^2.$$

Definition 5. *We say that the array (α) majorizes array (α'), and we write that in the following way $(\alpha')\prec(\alpha)$, if we can arrange the elements of arrays (α) and (α') in such a way that the following three conditions are satisfied:*

1. *$\alpha_1'+\alpha_2'+\dots+\alpha_n'=\alpha_1+\alpha_2+\dots+\alpha_n$;*
2. *$\alpha_1'\geq\alpha_2'\geq\dots\geq\alpha_n'$ i $\alpha_1\geq\alpha_2\geq\dots\geq\alpha_n$.*
3. *$\alpha_1'+\alpha_2'+\dots+\alpha_\nu'\leq\alpha_1+\alpha_2+\dots+\alpha_\nu$, for all $1\leq\nu<n$.*

Clearly, $(\alpha)\prec(\alpha)$.

Theorem 13 (Muirhead). *The necessairy and sufficient condition for comparability of $T[\alpha]$ and $T[\alpha']$, for all positive arrays (a), is that one of the arrays (α) and (α') majorizes the other. If $(\alpha')\prec(\alpha)$ then*

$$T[\alpha']\leq T[\alpha].$$

Equality holds if and only if (α) and (α') are identical, or when all a_is are equal.

Proof. First, we prove the necessity of the condition. Setting that all elements of the array a are equal to x, we get that

$$x^{\sum \alpha_i'} \leq x^{\sum \alpha_i}.$$

This can be satisfied for both large and small xs only if the condition 1 from the definition is satisfied. Now we put $a_1 = \cdots, a_\nu = x$ and $a_{\nu+1} = \cdots = a_n = 1$. Comparing the highest powers of x in expressions $T[\alpha]$ and $T[\alpha']$, knowing that for sufficiently large x we must have $T[\alpha'] \leq T[\alpha]$, we conclude that $\alpha_1' + \cdots + \alpha_\nu' \leq \alpha_1 + \cdots + \alpha_\nu$.

Now we will proof the sufficiency of the condition. The statement will follow from the following two lemmas. We will define one linear operation L on the set of the exponents (α). Suppose that α_k and α_l are two different exponents of (α) such that $\alpha_k > \alpha_l$. We can write

$$\alpha_k = \rho + \tau, \quad \alpha_l = \rho - \tau \quad (0 < \tau \leq \rho).$$

If $0 \leq \sigma < \tau \leq \rho$, define the array $(\alpha') = L(\alpha)$ in the following way:

$$\begin{cases} \alpha_k' = \rho + \sigma = \frac{\tau+\sigma}{2\tau}\alpha_k + \frac{\tau-\sigma}{2\tau}\alpha_l, \\ \alpha_l' = \rho - \sigma = \frac{\tau-\sigma}{2\tau}\alpha_k + \frac{\tau+\sigma}{2\tau}\alpha_l, \\ \alpha_\nu' = \alpha_\nu, \quad (\nu \neq k, \nu \neq l). \end{cases}$$

The definition of this mapping doesn't require that some of the arrays (α) and (α') is in non-decreasing order.

Lemma 4. *If $(\alpha') = L(\alpha)$, then $T[\alpha'] \leq T[\alpha]$, and equality holds if and only if all the elements of (a) are equal.*

Proof. We may rearrange the elements of the sequence such that $k = 1$ i $l = 2$. Then we have

$$\begin{aligned} & T[\alpha] - T[\alpha'] \\ = \; & \sum! a_3^{\alpha_3} \cdots a_n^{\alpha_n} \cdot (a_1^{\rho+\tau} a_2^{\rho-\tau} + a_1^{\rho-\tau} a_2^{\rho+\tau} - a_1^{\rho+\sigma} a_2^{\rho-\sigma} - a_1^{\rho-\sigma} a_2^{\rho+\sigma}) \\ = \; & \sum! (a_1 a_2)^{\rho-\tau} a_3^{\alpha_3} \cdots a_n^{\alpha_n} (a_1^{\tau+\sigma} - a_2^{\tau+\sigma})(a_1^{\tau-\sigma} - a_2^{\tau-\sigma}) \geq 0. \end{aligned}$$

Eaquality holds if and only if a_is are equal. □

Lemma 5. *If $(\alpha') \prec (\alpha)$, but (α') and (α) are different, then (α') can be obtained from (α) by succesive application of the transformation L.*

Proof. Denote by m the number of differences $\alpha_\nu - \alpha_\nu'$ that are $\neq 0$. m is a positive integer and we will prove that we can apply operation L in such a way that after each of applications, number m decreases (this would imply that the procedure will end up after finite number of steps). Since $\sum(\alpha_\nu - \alpha_\nu') = 0$, and not all of differences are 0, there are positive and negative differences, but the first one is positive. We can find such k and l for which:

$$\alpha_k' < \alpha_k, \quad \alpha_{k+1}' = \alpha_{k+1}, \ldots, \alpha_{l-1}' = \alpha_{l-1}, \quad \alpha_l' > \alpha_l.$$

($\alpha_l - \alpha_l'$ is the first negative difference, and $\alpha_k - \alpha_k'$ is the last positive difference before this negative one). Let $\alpha_k = \rho + \tau$ and $\alpha_l = \rho - \tau$, define σ by

$$\sigma = \max\{|\alpha_k' - \rho|, |\alpha_l' - \rho|\}.$$

At least one of the following two equalities is satisfied:

$$\alpha_l' - \rho = -\sigma, \quad \alpha_k' - \rho = \sigma,$$

because $\alpha'_k > \alpha'_l$. We also have $\sigma < \tau$, because $\alpha'_k < \alpha_k$ i $\alpha'_l > \alpha_l$. Let

$$\alpha''_k = \rho + \sigma, \quad \alpha''_l = \rho - \sigma, \quad \alpha''_\nu = \alpha_\nu \quad (\nu \neq k, \nu \neq l).$$

Now instead of the sequence (α) we will consider the sequence (α''). Number m has decreased by at least 1. It is easy to prove that the sequence (α'') is increasing and majorizes (α'). Repeating this procedure, we will get the sequence (α') which completes the proof of the second lemma, and hence the Muirhead's theorem. □ □

Example 6. *AM-GM is now the consequence of the Muirhead's inequality.*

Problem 9. *Prove that for positive numbers a, b and c the following equality holds:*

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}.$$

Solution. After multiplying both left and right-hand side of the required inequality with $abc(a^3 + b^3 + abc)(b^3 + c^3 + abc)(c^3 + a^3 + abc)$ we get that the original inequality is equivalent to

$$\begin{aligned} &\tfrac{3}{2}T[4,4,1] + 2T[5,2,2] + \tfrac{1}{2}T[7,1,1] + \tfrac{1}{2}T[3,3,3] \leq \\ &\leq \tfrac{1}{2}T[3,3,3] + T[6,3,0] + \tfrac{3}{2}T[4,4,1] + \tfrac{1}{2}T[7,1,1] + T[5,2,2] \end{aligned}$$

which is true because Muirhead's theorem imply that $T[5,2,2] \leq T[6,3,0]$. △

More problems with solutions using Muirhead's inequality can be found in the section "Problems".

5 Inequalities of Jensen and Karamata

Theorem 14 (Jensen's Inequality)**.** *If f is convex function and $\alpha_1, \dots, \alpha_n$ sequence of real numbers such that $\alpha_1 + \cdots + \alpha_n = 1$, than for any sequence $x_1, \dots, x_n$ of real numbers, the following inequality holds:*

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n).$$

Remark. If f is concave, then $f(\alpha_1 x_1 + \cdots + \alpha_n x_n) \geq \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n)$.

Example 7. *Using Jensen's inequality prove the generalized mean inequality, i.e. that for every two sequences of positive real numbers $x_1, \dots, x_n$ and $m_1, \dots, m_n$ such that $m_1 + \cdots + m_n = 1$ the following inequality holds:*

$$m_1 x_1 + m_2 x_2 + \cdots + m_n x_n \geq x_1^{m_1} \cdot x_2^{m_2} \cdots x_n^{m_n}.$$

Theorem 15 (Karamata's inequalities)**.** *Let f be a convex function and $x_1, \dots, x_n$, $y_1, y_2, \dots, y_n$ two non-increasing sequences of real numbers. If one of the following two conditions is satisfied:*

(a) $(y) \prec (x)$;

(b) $x_1 \geq y_1$, $x_1 + x_2 \geq y_1 + y_2$, $x_1 + x_2 + x_3 \geq y_1 + y_2 + y_3$, $\dots$, $x_1 + \cdots + x_{n-1} \geq y_1 + \cdots + y_{n-1}$, $x_1 + \cdots + x_n \geq y_1 + \cdots + y_n$ *and f is increasing;*

then

$$\sum_{i=1}^{n} f(x_i) \geq \sum_{i=1}^{n} f(y_i). \qquad (20)$$

Proof. Let $c_i = \frac{f(y_i)-f(x_i)}{y_i-x_i}$, for $y_i \neq x_i$, and $c_i = f'_+(x_i)$, for $x_i = y_i$. Since f is convex, and x_i, y_i are decreasing sequences, c_i is non-increasing (because is represents the "slope" of f on the interval between x_i and y_i). We now have

$$\begin{aligned}\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f(y_i) &= \sum_{i=1}^{n} c_i(x_i - y_i) = \sum_{i=1}^{n} c_i x_i - \sum_{i=1}^{n} c_i y_i \\ &= \sum_{i=1}^{n} (c_i - c_{i+1})(x_1 + \cdots + x_i) \\ &\quad - \sum_{i=1}^{n} (c_i - c_{i+1})(y_1 + \cdots + y_i), \qquad (21)\end{aligned}$$

here we define c_{n+1} to be 0. Now, denoting $A_i = x_1 + \cdots + x_i$ and $B_i = y_1 + \cdots + y_i$ (21) can be rearranged to

$$\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f(y_i) = \sum_{i=1}^{n-1} (c_i - c_{i+1})(A_i - B_i) + c_n \cdot (A_n - B_n).$$

The sum on the right-hand side of the last inequality is non-negative because c_i is decreasing and $A_i \geq B_i$. The last term $c_n(A_n - B_n)$ is zero under the assumption (a). Under the assumption (b) we have that $c_n \geq 0$ (f is increasing) and $A_n \geq B_n$ and this implies (20). □

Problem 10. *If $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ are two sequences of positive real numbers which satisfy the following conditions:*

$$a_1 \geq b_2,\ a_1a_2 \geq b_1b_2,\ a_1a_2a_3 \geq b_1b_2b_3,\ \cdots \geq a_1a_2\cdots a_n \geq b_1b_2\cdots b_n,$$

prove that

$$a_1 + a_2 + \cdots + a_n \geq b_1 + b_2 + \cdots + b_n.$$

Solution. Let $a_i = e^{x_i}$ and $b_i = e^{y_i}$. We easily verify that the conditions (b) of the Karamata's theorem are satisfied. Thus $\sum_{i=1}^{n} e^{y_i} \geq \sum_{i=1}^{n} e^{x_i}$ and the result immediately follows. △

Problem 11. *If $x_1, \ldots, x_n \in [-\pi/6, \pi/6]$, prove that*

$$\cos(2x_1 - x_2) + \cos(2x_2 - x_3) + \cdots + \cos(2x_n - x_1) \leq \cos x_1 + \cdots + \cos x_n.$$

Solution. Rearrange $(2x_1 - x_2, 2x_2 - x_3, \ldots, 2x_n - x_1)$ and $(x_1, \ldots, x_n)$ in two non-increasing sequences $(2x_{m_1} - x_{m_1+1}, 2x_{m_2} - x_{m_2+1}, \ldots, 2x_{m_n} - x_{m_n+1})$ and $(x_{k_1}, x_{k_2}, \ldots, x_{k_n})$ (here we assume that $x_{n+1} = x_1$. We will verify that condition (a) of the Karamata's inequality is satisfied. This follows from

$$\begin{aligned}&(2x_{m_1} - x_{m_1+1} + \cdots + 2x_{m_l} - x_{m_l+1}) - (x_{k_1} + \cdots + x_{k_l}) \\ \geq\ &(2x_{k_1} - x_{k_1+1} + \cdots + 2x_{k_l} - x_{k_l+1}) - (x_{k_1} + \cdots + x_{k_l}) \\ =\ &(x_{k_1} + \cdots x_{k_l}) - (x_{k_1+1} + \cdots + x_{k_l+1}) \geq 0.\end{aligned}$$

The function $f(x) = -\cos x$ is convex on $[-\pi/2, \pi/2]$ hence Karamata's inequality holds and we get

$$-\cos(2x_1 - x_2) - \cdots - \cos(2x_n - x_1) \geq -\cos x_1 - \cdots - \cos x_n,$$

which is obviously equivalent to the required inequality. △

6 Chebyshev's inequalities

Theorem 16 (Chebyshev's inequalities)**.** *Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ be real numbers. Then*

$$n\sum_{i=1}^{n} a_i b_i \geq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \geq n\sum_{i=1}^{n} a_i b_{n+1-i}. \tag{22}$$

The two inequalities become equalities at the same time when $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.

The Chebyshev's inequality will follow from the following generalization (placing $m_i = \frac{1}{n}$ for the left part, and the right inequality follows by applying the left on a_i and $c_i = -b_{n+1-i}$).

Theorem 17 (Generalized Chebyshev's Inequality)**.** *Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ be any real numbers, and $m_1, \dots, m_n$ non-negative real numbers whose sum is 1. Then*

$$\sum_{i=1}^{n} a_i b_i m_i \geq \left(\sum_{i=1}^{n} a_i m_i\right)\left(\sum_{i=1}^{n} b_i m_i\right). \tag{23}$$

The inequality become an equality if and only if $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.

Proof. From $(a_i - a_j)(b_i - b_j) \geq 0$ we get:

$$\sum_{i,j}(a_i - a_j)(b_i - b_j)m_i m_j \geq 0. \tag{24}$$

Since $(\sum_{i=1}^{n} a_i m_i)\cdot(\sum_{i=1}^{n} b_i m_i) = \sum_{i,j} a_i b_j m_i m_j$, (24) implies that

$$\begin{aligned} 0 &\leq \sum_{i,j} a_i b_i m_i m_j - \sum_{i,j} a_i b_j m_i m_j - \sum_{i,j} a_j b_i m_j m_i + \sum_{i,j} a_j b_j m_i m_j \\ &= 2\left[\sum_i a_i b_i m_i - \left(\sum_i a_i m_i\right)\left(\sum_i b_i m_i\right)\right]. \ \Box \end{aligned}$$

Problem 12. *Prove that the sum of distances of the orthocenter from the sides of an acute triangle is less than or equal to $3r$, where the r is the inradius.*

Solution. Denote $a = BC$, $b = CA$, $c = AB$ and let S_{ABC} denote the area of the triangle ABC. Let d_A, d_B, d_C be the distances from H to BC, CA, AB, and A', B', C' the feet of perpendiculars from A, B, C. Then we have $ad_a + bd_b + cd_c = 2(S_{BCH} + S_{ACH} + S_{ABH}) = 2P$. On the other hand if we assume that $a \geq b \geq c$, it is easy to prove that $d_A \geq d_B \geq d_C$. Indeed, $a \geq b$ implies $\angle A \geq \angle B$ hence $\angle HCB' \leq \angle HCA'$ and $HB' \leq HA'$. The Chebyshev's inequality implies

$$(a+b+c)r = 2P = ad_a + bd_b + cd_c \geqslant \frac{1}{3}(a+b+c)(d_a+d_b+d_c). \ \triangle$$

7 Problems

1. If $a, b, c, d > 0$, prove that

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b} \geq 2.$$

2. Prove that
$$\frac{a^3}{a^2+ab+b^2}+\frac{b^3}{b^2+bc+c^2}+\frac{c^3}{c^2+ca+a^2}\geq\frac{a+b+c}{3},$$
for $a,b,c>0$.

3. If $a,b,c,d,e,f>0$, prove that
$$\frac{ab}{a+b}+\frac{cd}{c+d}+\frac{ef}{e+f}\leq\frac{(a+c+e)(b+d+f)}{a+b+c+d+e+f}.$$

4. If $a,b,c\geq 1$, prove that
$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1}\leq\sqrt{c(ab+1)}.$$

5. Let $a_1,a_2,\dots,a_n,b_1,b_2,\dots,b_n$ be positive real numbers. Prove that
$$\left(\sum_{i\neq j}a_ib_j\right)^2\geq\left(\sum_{i\neq j}a_ia_j\right)\left(\sum_{i\neq j}b_ib_j\right).$$

6. If $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ for $x,y,z>0$, prove that
$$(x-1)(y-1)(z-1)\geq 8.$$

7. Let $a,b,c>0$ satisfy $abc=1$. Prove that
$$\frac{1}{\sqrt{b+\frac{1}{a}+\frac{1}{2}}}+\frac{1}{\sqrt{c+\frac{1}{b}+\frac{1}{2}}}+\frac{1}{\sqrt{a+\frac{1}{c}+\frac{1}{2}}}\geq\sqrt{2}.$$

8. Given positive numbers a,b,c,x,y,z such that $a+x=b+y=c+z=S$, prove that $ay+bz+cx<S^2$.

9. Let a,b,c be positive real numbers. Prove the inequality
$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq a+b+c+\frac{4(a-b)^2}{a+b+c}.$$

10. Determine the maximal real number a for which the inequality
$$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2\geq a(x_1x_2+x_2x_3+x_3x_4+x_4x_5)$$
holds for any five real numbers x_1,x_2,x_3,x_4,x_5.

11. If $x,y,z\geq 0$ and $x+y+z=1$, prove that
$$0\leq xy+yz+zx-2xyz\leq\frac{7}{27}.$$

12. Let a,b and c be positive real numbers such that $abc=1$. Prove that
$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$

13. If a, b and c are positive real numbers, prove that:

$$\frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ca+a^2}+\frac{c^3}{a^2-ab+b^2}\geq 3\cdot\frac{ab+bc+ca}{a+b+c}.$$

14. (IMO05) Let x, y and z be positive real numbers such that $xyz \geq 1$. Prove that

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2}\geq 0.$$

15. Let $a_1, \dots, a_n$ be positive real numbers. Prove that

$$\frac{a_1^3}{a_2}+\frac{a_2^3}{a_3}+\cdots+\frac{a_n^3}{a_1}\geq a_1^2+a_2^2+\cdots+a_n^2.$$

16. Let $a_1, \dots, a_n$ be positive real numbers. Prove that

$$(1+a_1)(1+a_2)\cdots(1+a_n)\leq\left(1+\frac{a_1^2}{a_2}\right)\cdot\left(1+\frac{a_2^2}{a_3}\right)\cdot\cdots\cdot\left(1+\frac{a_n^2}{a_1}\right).$$

17. If a, b, and c are the lengths of the sides of a triangle, s its semiperimeter, and $n \geq 1$ an integer, prove that

$$\frac{a^n}{b+c}+\frac{b^n}{c+a}+\frac{c^n}{a+b}\geq\left(\frac{2}{3}\right)^{n-2}\cdot s^{n-1}.$$

18. Let $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ ($n \geq 2$) and

$$\frac{1}{1+x_1}+\frac{1}{1+x_2}+\cdots+\frac{1}{1+x_n}=1.$$

Prove that

$$\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n}\geq(n-1)\left(\frac{1}{\sqrt{x_1}}+\frac{1}{\sqrt{x_2}}+\cdots+\frac{1}{\sqrt{x_n}}\right).$$

19. Suppose that any two members of certain society are either *friends* or *enemies*. Suppose that there is total of n members, that there is total of q pairs of friends, and that in any set of three persons there are two who are enemies to each other. Prove that there exists at least one member among whose enemies we can find at most $q\cdot\left(1-\frac{4q}{n^2}\right)$ pairs of friends.

20. Given a set of unit circles in the plane whose total area is S. Prove that among those circles there exist certain number of non-intersecting circles whose total area is $\geq \frac{2}{9}S$.

8 Solutions

1. Denote by L the left-hand side of the required inequality. If we add the first and the third summand of L we get

$$\frac{a}{b+c}+\frac{c}{d+a}=\frac{a^2+c^2+ad+bc}{(b+c)(a+d)}.$$

We will bound the denominator of the last fraction using the inequality $xy \leq (x+y)^2/4$ for appropriate x and y. For $x = b+c$ and $y = a+d$ we get $(b+c)(a+d) \leq (a+b+c+d)^2/4$. The equality holds if and only if $a+d = b+c$. Therefore

$$\frac{a}{b+c} + \frac{c}{d+a} \geq 4\frac{a^2+c^2+ad+bc}{(a+b+c+d)^2}.$$

Similarly $\frac{b}{c+d} + \frac{d}{a+b} \geq 4\frac{b^2+d^2+ab+cd}{(a+b+c+d)^2}$ (with the equality if and only if $a+b = c+d$) implying

$$\begin{aligned} & \frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b} \\ \geq\ & 4\frac{a^2+b^2+c^2+d^2+ad+bc+ab+cd}{(a+b+c+d)^2} \\ =\ & 4\frac{a^2+b^2+c^2+d^2+(a+c)(b+d)}{[(a+c)+(b+d)]^2}. \end{aligned}$$

In order to solve the problem it is now enough to prove that

$$2\frac{a^2+b^2+c^2+d^2+(a+c)(b+d)}{[(a+c)+(b+d)]^2} \geq 1. \tag{25}$$

After multiplying both sides of (25) by $[(a+c)+(b+d)]^2 = (a+c)^2+(b+d)^2$ it becomes equivalent to $2(a^2+b^2+c^2+d^2) \geq (a+c)^2+(b+d)^2 = a^2+b^2+c^2+d^2+2ac+2bd$. It is easy to see that the last inequality holds because many terms will cancel and the remaining inequality is the consequence of $a^2+c^2 \geq 2ac$ and $b^2+d^2 \geq 2bc$. The equality holds if and only if $a = c$ and $b = d$.

2. We first notice that

$$\frac{a^3-b^3}{a^2+ab+b^2} + \frac{b^3-c^3}{b^2+bc+c^2} + \frac{c^3-a^3}{c^2+ca+a^2} = 0.$$

Hence it is enough to prove that

$$\frac{a^3+b^3}{a^2+ab+b^2} + \frac{b^3+c^3}{b^2+bc+c^2} + \frac{c^3+a^3}{c^2+ca+a^2} \geq \frac{2(a+b+c)}{3}.$$

However since $3(a^2-ab+b^2) \geq a^2+ab+b^2$,

$$\frac{a^3+b^3}{a^2+ab+b^2} = (a+b)\frac{a^2-ab+b^2}{a^2+ab+b^2} \geq \frac{a+b}{3}.$$

The equality holds if and only if $a = b = c$.

Second solution. First we prove that

$$\frac{a^3}{a^2+ab+b^2} \geq \frac{2a-b}{3}. \tag{26}$$

Indeed after multiplying we get that the inequality is equivalent to $a^3+b^3 \geq ab(a+b)$, or $(a+b)(a-b)^2 \geq 0$ which is true. After adding (26) with two similar inequalities we get the result.

3. We will first prove that
$$\frac{ab}{a+b}+\frac{cd}{c+d}\leq\frac{(a+c)(b+d)}{a+b+c+d}. \tag{27}$$
As is the case with many similar inequalities, a first look at (27) suggests to multiply out both sides by $(a+b)(c+d)(a+b+c+d)$. That looks scary. But we will do that now. In fact you will do, I will not. I will just encourage you and give moral support (try to imagine me doing that). After you multiply out everything (do it twice, to make sure you don't make a mistake in calculation), the result will be rewarding. Many things cancel out and what remains is to verify the inequality $4abcd\leq a^2d^2+b^2c^2$ which is true because it is equivalent to $0\leq(ad-bc)^2$. The equality holds if and only if $ad=bc$, or $\frac{a}{b}=\frac{c}{d}$.

   Applying (27) with the numbers $A=a+c, B=b+d, C=e$, and $D=f$ yields:
$$\frac{(a+c)(b+d)}{a+b+c+d}+\frac{ef}{e+f}\leq\frac{(A+C)(B+D)}{A+B+C+D}=\frac{(a+c+e)(b+d+f)}{a+b+c+d+e+f},$$
and the required inequality is proved because (27) can be applied to the first term of the left-hand side. The equality holds if and only if $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$.

4. To prove the required inequality we will use the similar approach as in the previous problem. First we prove that
$$\sqrt{a-1}+\sqrt{b-1}\leq\sqrt{ab}. \tag{28}$$
Squaring both sides gives us that the original inequality is equivalent to
$$\begin{aligned} & a+b-2+2\sqrt{(a-1)(b-1)}\leq ab \\ \Leftrightarrow\quad & 2\sqrt{(a-1)(b-1)}\leq ab-a-b+2=(a-1)(b-1)+1. \end{aligned} \tag{29}$$
The inequality (29) is true because it is of the form $x+1\geq2\sqrt{x}$ for $x=(a-1)(b-1)$.

   Now we will apply (28) on numbers $A=ab+1$ and $B=c$ to get
$$\sqrt{ab}+\sqrt{c-1}=\sqrt{A-1}+\sqrt{B-1}\leq\sqrt{AB}=\sqrt{(ab+1)c}.$$
The first term of the left-hand side is greater than or equal to $\sqrt{a-1}+\sqrt{b-1}$ which proves the statement. The equality holds if and only if $(a-1)(b-1)=1$ and $ab(c-1)=1$.

5. Let us denote $p=\sum_{i=1}^n a_i, q=\sum_{i=1}^n b_i, k=\sum_{i=1}^n a_i^2, l=\sum_{i=1}^n b_i^2$, and $m=\sum_{i=1}^n a_ib_i$. The following equalities are easy to verify:
$$\sum_{i\neq j}a_ib_j=pq-m,\ \sum_{i\neq j}a_ia_j=p^2-k,\text{ and }\sum_{i\neq j}b_ib_j=q^2-l,$$
so the required inequality is equivalent to
$$(pq-m)^2\geq(p^2-k)(q^2-l)\Leftrightarrow lp^2-2qm\cdot p+m^2+q^2k-kl\geq0.$$
Consider the last expression as a quadratic equation in p, i.e. $\varphi(p)=lp^2-2qm\cdot p+q^2k-kl$. If we prove that its discriminant is less than or equal to 0, we are done. That condition can be written as:
$$q^2m^2-l(m^2+q^2k-kl)\leq0\Leftrightarrow(lk-m^2)(q^2-l)\geq0.$$

The last inequality is true because $q^2 - l = \sum_{i \neq j} b_i b_j > 0$ (b_i are positive), and $lk - m^2 \geq 0$ (Cauchy-Schwartz inequality). The equality holds if and only if $lk - m^2 = 0$, i.e. if the sequences (a) and (b) are proportional.

6. This is an example of a problem where we have some conditions on x, y, and z. Since there are many reciprocals in those conditions it is natural to divide both sides of the original inequality by xyz. Then it becomes
$$\left(1 - \frac{1}{x}\right) \cdot \left(1 - \frac{1}{y}\right) \cdot \left(1 - \frac{1}{z}\right) \geq \frac{8}{xyz}. \tag{30}$$
However $1 - \frac{1}{x} = \frac{1}{y} + \frac{1}{z}$ and similar relations hold for the other two terms of the left-hand side of (30). Hence the original inequality is now equivalent to
$$\left(\frac{1}{y} + \frac{1}{z}\right) \cdot \left(\frac{1}{z} + \frac{1}{x}\right) \cdot \left(\frac{1}{x} + \frac{1}{y}\right) \geq \frac{8}{xyz},$$
and this follows from $\frac{1}{x} + \frac{1}{y} \geq 2\frac{1}{\sqrt{xy}}$, $\frac{1}{y} + \frac{1}{z} \geq 2\frac{1}{\sqrt{yz}}$, and $\frac{1}{z} + \frac{1}{x} \geq 2\frac{1}{\sqrt{zx}}$. The equality holds if and only if $x = y = z = 3$.

7. Notice that
$$\frac{1}{2} + b + \frac{1}{a} + \frac{1}{2} > 2\sqrt{\frac{1}{2} \cdot \left(b + \frac{1}{a} + \frac{1}{2}\right)}.$$
This inequality is strict for any two positive numbers a and b. Using the similar inequalities for the other two denominators on the left-hand side of the required inequality we get:
$$\begin{aligned} & \frac{1}{\sqrt{b + \frac{1}{a} + \frac{1}{2}}} + \frac{1}{\sqrt{c + \frac{1}{b} + \frac{1}{2}}} + \frac{1}{\sqrt{a + \frac{1}{c} + \frac{1}{2}}} \\ > \quad & \sqrt{2}\left(\frac{1}{1 + \frac{1}{a} + b} + \frac{1}{1 + \frac{1}{b} + c} + \frac{1}{1 + \frac{1}{c} + a}\right). \end{aligned} \tag{31}$$
The last expression in (31) can be transformed using $\frac{1}{1+\frac{1}{a}+b} = \frac{a}{1+a+ab} = \frac{a}{1+\frac{1}{c}+a}$ and $\frac{1}{1+\frac{1}{b}+c} = \frac{1}{c(ab+a+1)} = \frac{\frac{1}{c}}{1+\frac{1}{c}+a}$. Thus
$$\begin{aligned} & \sqrt{2}\left(\frac{1}{1 + \frac{1}{a} + b} + \frac{1}{1 + \frac{1}{b} + c} + \frac{1}{1 + \frac{1}{c} + a}\right) \\ = \quad & \sqrt{2} \cdot \frac{1 + \frac{1}{c} + a}{1 + \frac{1}{c} + a} = \sqrt{2}. \end{aligned}$$
The equality can never hold.

8. Denote $T = S/2$. One of the triples (a,b,c) and (x,y,z) has the property that at least two of its members are greater than or equal to T. Assume that (a,b,c) is the one, and choose $\alpha = a - T$, $\beta = b - T$, and $\gamma = c - T$. We then have $x = T - \alpha$, $y = T - \beta$, and $z = T - \gamma$. Now the required inequality is equivalent to
$$(T + \alpha)(T - \beta) + (T + \beta)(T - \gamma) + (T + \gamma)(T - \alpha) < 4T^2.$$

After simplifying we get that what we need to prove is

$$-(\alpha\beta+\beta\gamma+\gamma\alpha)<T^2. \tag{32}$$

We also know that at most one of the numbers α, β, γ is negative. If all are positive, there is nothing to prove. Assume that $\gamma<0$. Now (32) can be rewritten as $-\alpha\beta-\gamma(\alpha+\beta)<T^2$. Since $-\gamma<T$ we have that $-\alpha\beta-\gamma(\alpha+\beta)<-\alpha\beta+T(\alpha+\beta)$ and the last term is less than T since $(T-\alpha)(T-\beta)>0$.

9. Starting from $\frac{(a-b)^2}{b}=\frac{a^2}{b}-2a+b$ and similar equalitites for $(b-c)^2/c$ and $(c-a)^2/a$ we get the required inequality is equivalent to

$$(a+b+c)\left(\frac{(a-b)^2}{b}+\frac{(b-c)^2}{a}+\frac{(c-a)^2}{b}\right)\geq 4(a-b)^2. \tag{33}$$

By the Cauchy-Schwartz inequality we have that the left-hand side of (33) is greater than or equal to $(|a-b|+|b-c|+|c-a|)^2$. (33) now follows from $|b-c|+|c-a|\geq|a-b|$.

10. Note that

$$\begin{aligned} & x_1^2+x_2^2+x_3^2+x_4^2+x_5^2 \\ = & \left(x_1^2+\frac{x_2^2}{3}\right)+\left(\frac{2x_2^2}{3}+\frac{x_3^2}{2}\right)+\left(\frac{x_3^2}{2}+\frac{2x_4^2}{3}\right)+\left(\frac{x_4^2}{3}+x_5^2\right). \end{aligned}$$

Now applying the inequality $a^2+b^2\geq 2ab$ we get

$$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2\geq\frac{2}{\sqrt{3}}(x_1x_2+x_2x_3+x_3x_4+x_4x_5).$$

This proves that $a\geq\frac{2}{\sqrt{3}}$. In order to prove the other inequality it is sufficient to notice that for $(x_1,x_2,x_3,x_4,x_5)=(1,\sqrt{3},2,\sqrt{3},1)$ we have

$$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2=\frac{2}{\sqrt{3}}(x_1x_2+x_2x_3+x_3x_4+x_4x_5).$$

11. Since $xy+yz+zx-2xyz=(x+y+z)(xy+yz+zx)-2xyz=T[2,1,0]+\frac{1}{6}T[1,1,1]$ the left part of the inequality follows immediately. In order to prove the other part notice that

$$\frac{7}{27}=\frac{7}{27}(x+y+z)^3=\frac{7}{27}\left(\frac{1}{2}T[3,0,0]+3T[2,1,0]+T[1,1,1]\right).$$

After multiplying both sides by 54 and cancel as many things as possible we get that the required inequality is equivalent to:

$$12T[2,1,0]\leq 7T[3,0,0]+5T[1,1,1].$$

This inequality is true because it follows by adding up the inequalities $2T[2,1,0]\leq 2T[3,0,0]$ and $10T[2,1,0]\leq 5T[3,0,0]+5T[1,1,1]$ (the first one is a consequence of the Muirhead's and the second one of the Schur's theorem for $\alpha=\beta=1$).

12. The expressions have to be homogenous in order to apply the Muirhead's theorem. First we divide both left and right-hand side by $(abc)^{\frac{4}{3}} = 1$ and after that we multiply both sides by $a^3b^3c^3(a+b)(b+c)(c+a)(abc)^{\frac{4}{3}}$. The inequality becomes equivalent to

$$2T\left[\frac{16}{3},\frac{13}{3},\frac{7}{3}\right]+T\left[\frac{16}{3},\frac{16}{3},\frac{4}{3}\right]+T\left[\frac{13}{3},\frac{13}{3},\frac{10}{3}\right]\geq 3T[5,4,3]+T[4,4,4].$$

The last inequality follows by adding the following three which are immediate consequences of the Muirhead's theorem:

1. $2T\left[\frac{16}{3},\frac{13}{3},\frac{7}{3}\right]\geq 2T[5,4,3]$,
2. $T\left[\frac{16}{3},\frac{16}{3},\frac{4}{3}\right]\geq T[5,4,3]$,
3. $T\left[\frac{13}{3},\frac{13}{3},\frac{10}{3}\right]\geq T[4,4,4]$.

The equality holds if and only if $a=b=c=1$.

13. The left-hand side can be easily transformed into $\frac{a^3(b+c)}{b^3+c^3}+\frac{b^3(c+a)}{c^3+a^3}+\frac{c^3(a+b)}{a^3+b^3}$. We now multiply both sides by $(a+b+c)(a^3+b^3)(b^3+c^3)(c^3+a^3)$. After some algebra the left-hand side becomes

$$\begin{aligned}L = \;& T[9,2,0]+T[10,1,0]+T[9,1,1]+T[5,3,3]+2T[4,4,3]\\ &+T[6,5,0]+2T[6,4,1]+T[6,3,2]+T[7,4,0]+T[7,3,1],\end{aligned}$$

while the right-hand side transforms into

$$D = 3(T[4,4,3]+T[7,4,0]+T[6,4,1]+T[7,3,1]).$$

According to Muirhead's theorem we have:

1. $T[9,2,0]\geq T[7,4,0]$,
2. $T[10,1,0]\geq T[7,4,0]$,
3. $T[6,5,0]\geq T[6,4,1]$,
4. $T[6,3,2]\geq T[4,4,3]$.

The Schur's inequality gives us $T[4,2,2]+T[8,0,0]\geq 2T[6,2,0]$. After multiplying by abc, we get:

5. $T[5,3,3]+T[9,1,1]\geq T[7,3,1]$.

Adding up $1,2,3,4$, 5, and adding $2T[4,4,3]+T[7,4,0]+2T[6,4,1]+T[7,3,1]$ to both sides we get $L\geq D$. The equality holds if and only if $a=b=c$.

14. Multiplying the both sides with the common denominator we get

$$T_{5,5,5}+4T_{7,5,0}+T_{5,2,2}+T_{9,0,0}\geq T_{5,5,2}+T_{6,0,0}+2T_{5,4,0}+2T_{4,2,0}+T_{2,2,2}.$$

By Schur's and Muirhead's inequalities we have that $T_{9,0,0}+T_{5,2,2}\geq 2T_{7,2,0}\geq 2T_{7,1,1}$. Since $xyz\geq 1$ we have that $T_{7,1,1}\geq T_{6,0,0}$. Therefore

$$T_{9,0,0}+T_{5,2,2}\geq 2T_{6,0,0}\geq T_{6,0,0}+T_{4,2,0}.$$

Moreover, Muirhead's inequality combined with $xyz\geq 1$ gives us $T_{7,5,0}\geq T_{5,5,2}$, $2T_{7,5,0}\geq 2T_{6,5,1}\geq 2T_{5,4,0}$, $T_{7,5,0}\geq T_{6,4,2}\geq T_{4,2,0}$, and $T_{5,5,5}\geq T_{2,2,2}$. Adding these four inequalities to (1) yields the desired result.

15. Let $a_i = e^{x_i}$ and let $(m_1,\dots,m_n)$, $(k_1,\dots,k_n)$ be two permutations of $(1,\dots,n)$ for which the sequences $(3x_{m_1} - x_{m_1+1},\dots,3x_{m_n} - x_{m_n+1})$ and $(2x_{k_1},\dots,2x_{k_n})$ are non-increasing. As above we assume that $x_{n+1} = x_n$. Similarly as in the problem 11 from the section 5 we prove that $(2x_{k_i}) \prec (3x_{m_i} - x_{m_i+1})$. The function $f(x) = e^x$ is convex so the Karamata's implies the required result.

16. Hint: Choose x_i such that $a_i = e^{x_i}$. Sort the sequences $(2x_1 - x_2,\dots,2x_n - x_1)$ and $(x_1,\dots,x_n)$ in non-increasing order, prove that the first majorizes the second, and apply Karamata's inequality with the convex function $f(x) = 1 + e^x$.

17. Applying the Chebyshev's inequality first we get

$$\frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} \geq \frac{a^n+b^n+c^n}{3} \cdot \left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right).$$

The Cauchy-Schwartz inequality gives:

$$2(a+b+c)\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \geq 9,$$

and the inequality $M_n \geq M_2$ gives

$$\frac{a^n+b^n+c^n}{3} \geq \left(\frac{a+b+c}{3}\right)^n.$$

In summary

$$\begin{aligned} \frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} &\geq \left(\frac{a+b+c}{3}\right)^n \left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \\ &\geq \frac{1}{3} \cdot \frac{1}{2} \cdot \left(\frac{2}{3}s\right)^{n-1} \cdot 9 = \left(\frac{2}{3}\right)^{n-2} s^{n-1}. \end{aligned}$$

18. It is enough to prove that

$$\begin{aligned} &\left(\sqrt{x_1} + \frac{1}{\sqrt{x_1}}\right) + \left(\sqrt{x_2} + \frac{1}{\sqrt{x_2}}\right) + \cdots + \left(\sqrt{x_n} + \frac{1}{\sqrt{x_n}}\right) \\ \geq\ & n\left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}}\right), \end{aligned}$$

or equivalently

$$\begin{aligned} &\left(\frac{1+x_1}{\sqrt{x_1}} + \cdots + \frac{1+x_n}{\sqrt{x_n}}\right)\left(\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n}\right) \\ \geq\ & n \cdot \left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}}\right). \end{aligned}$$

Consider the function $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}} = \frac{x+1}{\sqrt{x}}, x \in (0,+\infty)$. It is easy to verify that f is non-decreasing on$(1,+\infty)$ and that $f(x) = f\left(\frac{1}{x}\right)$ for every $x > 0$. Furthermore from the given conditions it follows that only x_1 can be less than 1 and that $\frac{1}{1+x_2} \le 1 - \frac{1}{1+x_1} = \frac{x_1}{1+x_1}$. Hence $x_2 \ge \frac{1}{x_1}$. Now it is clear that (in both of the cases $x_1 \ge 1$ and $x_1 < 1$):

$$f(x_1) = f\left(\frac{1}{x_1}\right) \le f(x_1) \le \cdots \le f(x_n).$$

This means that the sequence $\left(\frac{1+x_k}{x_k}\right)_{k=1}^{n}$ is non-decreasing. Thus according to the Chebyshev's inequality we have:

$$\begin{aligned} & \left(\frac{1+x_1}{\sqrt{x_1}} + \cdots + \frac{1+x_n}{\sqrt{x_n}}\right)\left(\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n}\right) \\ \ge \quad & n \cdot \left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}}\right). \end{aligned}$$

The equality holds if and only if $\frac{1}{1+x_1} = \cdots = \frac{1}{1+x_n}$, or $\frac{1+x_1}{\sqrt{x_1}} = \cdots = \frac{1+x_n}{\sqrt{x_n}}$, which implies that $x_1 = x_2 = \cdots = x_n$. Thus the equality holds if and only if $x_1 = \cdots = x_n = n-1$.

19. Denote by S the set of all members of the society, by A the set of all pairs of friends, and by N the set of all pairs of enemies. For every $x \in S$, denote by $f(x)$ number of friends of x and by $F(x)$ number of pairs of friends among enemies of x. It is easy to prove:

$$q = |A| = \frac{1}{2}\sum_{x \in S} f(x);$$

$$\sum_{\{a,b\} \in A} (f(a) + f(b)) = \sum_{x \in S} f^2(x).$$

If a and b are friends, then the number of their common enemies is equal to $(n-2) - (f(a) - 1) - (f(b) - 1) = n - f(a) - f(b)$. Thus

$$\frac{1}{n}\sum_{x \in S} F(x) = \frac{1}{n}\sum_{\{a,b\} \in A} (n - f(a) - f(b)) = q - \frac{1}{n}\sum_{x \in S} f^2(x).$$

Using the inequality between arithmetic and quadratic mean on the last expression, we get

$$\frac{1}{n}\sum_{x \in S} F(x) \le q - \frac{4q^2}{n^2}$$

and the statement of the problem follows immediately.

20. Consider the partition of plane π into regular hexagons, each having inradius 2. Fix one of these hexagons, denoted by γ. For any other hexagon x in the partition, there exists a unique translation τ_x taking it onto γ. Define the mapping $\varphi : \pi \to \gamma$ as follows: If A belongs to the interior of a hexagon x, then $\varphi(A) = \tau_x(A)$ (if A is on the border of some hexagon, it does not actually matter where its image is).

The total area of the images of the union of the given circles equals S, while the area of the hexagon γ is $8\sqrt{3}$. Thus there exists a point B of γ that is covered at least $\dfrac{S}{8\sqrt{3}}$ times, i.e., such that $\varphi^{-1}(B)$ consists of at least $\dfrac{S}{8\sqrt{3}}$ distinct points of the plane that belong to some of the circles. For any of these points, take a circle that contains it. All these circles are disjoint, with total area not less than $\dfrac{\pi}{8\sqrt{3}}S \geq 2S/9$.

Polynomial Equations

Dušan Djukić

Contents

1 Introduction

The title refers to determining polynomials in one or more variables (e.g. with real or complex coefficients) which satisfy some given relation(s).

The following example illustrates some basic methods:

1. Determine the polynomials P for which $16P(x^2) = P(2x)^2$.

- *First method: evaluating at certain points and reducing degree.*

  Plugging $x = 0$ in the given relation yields $16P(0) = P(0)^2$, i.e. $P(0) = 0$ or 16.

  (i) Suppose that $P(0) = 0$. Then $P(x) = xQ(x)$ for some polynomial Q and $16x^2Q(x^2) = 4x^2Q(2x)^2$, which reduces to $4Q(x^2) = Q(2x)^2$. Now setting $4Q(x) = R(x)$ gives us $16R(x^2) = R(2x)^2$. Hence, $P(x) = \frac{1}{4}xR(x)$, with R satisfying the same relation as P.

  (ii) Suppose that $P(0) = 16$. Putting $P(x) = xQ(x) + 16$ in the given relation we obtain $4xQ(x^2) = xQ(2x)^2 + 16Q(2x)$; hence $Q(0) = 0$, i.e. $Q(x) = xQ_1(x)$ for some polynomial Q_1. Furthermore, $x^2Q_1(x^2) = x^2Q_1(2x)^2 + 8Q_1(2x)$, implying that $Q_1(0) = 0$, so Q_1 too is divisible by x. Thus $Q(x) = x^2Q_1(x)$. Now suppose that x^n is the highest degree of x dividing Q, and $Q(x) = x^nR(x)$, where $R(0) \neq 0$. Then R satisfies $4x^{n+1}R(x^2) = 2^{2n}x^{n+1}R(2x)^2 + 2^{n+4}R(2x)$, which implies that $R(0) = 0$, a contradiction. It follows that $Q \equiv 0$ and $P(x) \equiv 16$.

  We conclude that $P(x) = 16\left(\frac{1}{4}x\right)^n$ for some $n \in \mathbb{N}_0$.

- *Second method: investigating coefficients.*

  We start by proving the following lemma (to be used frequently):

  Lemma 1. *If $P(x)^2$ is a polynomial in x^2, then so is either $P(x)$ or $P(x)/x$.*

  Proof. Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_n \neq 0$. The coefficient at x^{2n-1} is $2a_na_{n-1}$, from which we get $a_{n-1} = 0$. Now the coefficient at x^{2n-3} equals $2a_na_{n-3}$; hence $a_{n-3} = 0$, and so on. Continuing in this manner we conclude that $a_{n-2k-1} = 0$ for $k = 0, 1, 2, \dots$, i.e. $P(x) = a_nx^n + a_{n-2}x^{n-2} + a_{n-4}x^{n-4} + \cdots$. △

  Since $P(x)^2 = 16P(x^2/4)$ is a polynomial in x^2, we have $P(x) = Q(x^2)$ or $P(x) = xQ(x^2)$. In the former case we get $16Q(x^4) = Q(4x^2)^2$ and therefore $16Q(x^2) = Q(4x)^2$; in the latter case we similarly get $4Q(x^2) = Q(4x)^2$. In either case, $Q(x) = R(x^2)$ or $Q(x) = xR(x^2)$ for some polynomial R, so $P(x) = x^i R(x^4)$ for some $i \in \{0,1,2,3\}$. Proceeding in this way we find that $P(x) = x^i S(x^{2^k})$ for each $k \in \mathbb{N}$ and some $i \in \{0,1,\dots,2^k\}$. Now it is enough to take k with $2^k > \deg P$ and to conclude that S must be constant. Thus $P(x) = cx^i$ for some $c \in \mathbb{R}$. A simple verification gives us the general solution $P(x) = 16\left(\frac{1}{4}x\right)^n$ for $n \in \mathbb{N}_0$.

Investigating zeroes of the unknown polynomial is also counted under the first method.

A majority of problems of this type can be solved by one of the above two methods (although some cannot, making math more interesting!).

2 Problems with Solutions

1. Find all polynomials P such that $P(x)^2 + P(\frac{1}{x})^2 = P(x^2)P(\frac{1}{x^2})$.

   Solution. By the introducing lemma there exists a polynomial Q such that $P(x) = Q(x^2)$ or $P(x) = xQ(x^2)$. In the former case $Q(x^2)^2 + Q(\frac{1}{x^2}) = Q(x^4)Q(\frac{1}{x^4})$ and therefore $Q(x)^2 + Q(\frac{1}{x}) = Q(x^2)Q(\frac{1}{x^2})$ (which is precisely the relation fulfilled by P), whereas in the latter case (similarly) $xQ(x)^2 + \frac{1}{x}Q(\frac{1}{x})^2 = Q(x^2)Q(\frac{1}{x^2})$ which is impossible for the left and right hand side have odd and even degrees, respectively. We conclude that $P(x) = Q(x^2)$, where Q is also a solution of the considered polynomial equation. Considering the solution of the least degree we find that P must be constant.

2. Do there exist non-linear polynomials P and Q such that $P(Q(x)) = (x-1)(x-2)\cdots(x-15)$?

   Solution. Suppose there exist such polynomials. Then $\deg P \cdot \deg Q = 15$, so $\deg P = k \in \{3,5\}$. Putting $P(x) = c(x-a_1)\cdots(x-a_k)$ we get $c(Q(x)-a_1)\cdots(Q(x)-a_k) = (x-1)(x-2)\cdots(x-15)$. Thus the roots of polynomial $Q(x)-a_i$ are distinct and comprise the set $\{1,2,\dots,15\}$. All these polynomials mutually differ at the last coefficient only. Now, investigating parity of the remaining (three or five) coefficients we conclude that each of them has the equally many odd roots. This is impossible, since the total number of odd roots is 8, not divisible by 3 or 5.

3. Determine all polynomials P for which $P(x)^2 - 2 = 2P(2x^2-1)$.

   Solution. Denote $P(1) = a$. We have $a^2 - 2a - 2 = 0$. Substituting $P(x) = (x-1)P_1(x) + a$ in the initial relation and simplifying yields $(x-1)P_1(x)^2 + 2aP_1(x) = 4(x+1)P_1(2x^2-1)$. For $x = 1$ we have $2aP_1(1) = 8P_1(1)$, which (since $a \neq 4$) gives us $P_1(1) = 0$, i.e. $P_1(x) = (x-1)P_2(x)$, so $P(x) = (x-1)^2P_2(x) + a$. Suppose that $P(x) = (x-1)^nQ(x) + a$, where $Q(1) \neq 0$. Substituting in the initial relation and simplifying yields $(x-1)^nQ(x)^2 + 2aQ(x) = 2(2x+2)^nQ(2x^2-1)$, giving us $Q(1) = 0$, a contradiction. It follows that $P(x) = a$.

4. Determine all polynomials P for which $P(x)^2 - 1 = 4P(x^2-4x+1)$.

   Solution. Suppose that P is not constant. Fixing $\deg P = n$ and comparing coefficients of both sides we deduce that the coefficients of polynomial P must be rational. On the other hand, setting $x = a$ with $a = a^2 - 4a + 1$, that is, $a = \frac{5\pm\sqrt{21}}{2}$, we obtain $P(a) = b$, where $b^2 - 4b - 1 = 0$, i.e. $b = 2 \pm \sqrt{5}$. However, this is impossible because $P(a)$ must be of the form $p + q\sqrt{21}$ for some rational p, q for the coefficients of P are rational. It follows that $P(x)$ is constant.

5. For which real values of a does there exist a rational function $f(x)$ that satisfies $f(x^2) = f(x)^2 - a$?

   Solution. Write f as $f = P/Q$ with P and Q coprime polynomials and Q monic. By comparing leading coefficients we obtain that P too is monic. The condition of the problem became $P(x^2)/Q(x^2) = P(x)^2/Q(x)^2 - a$. Since $P(x^2)$ and $Q(x^2)$ are coprime (if, to the contrary, they had a zero in common, then so do P and Q), it follows that $Q(x^2) = Q(x)^2$. Therefore $Q(x) = x^n$ for some $n \in \mathbb{N}$. Now we have $P(x^2) = P(x)^2 - ax^{2n}$.

Let $P(x) = a_0 + a_1x + \cdots + a_{m-1}x^{m-1} + x^m$. Comparing coefficients of $P(x)^2$ and $P(x^2)$ gives us $a_{n-1} = \cdots = a_{2m-n+1} = 0$, $a_{2m-n} = a/2$, $a_1 = \cdots = a_{m-1} = 0$ and $a_0 = 1$. This is only possible if $a = 2$ and $2m - n = 0$, or $a = 0$.

6. Find all polynomials P satisfying $P(x^2+1) = P(x)^2 + 1$ for all x.

   Solution. By the introducing lemma, there is a polynomial Q such that $P(x) = Q(x^2+1)$ or $P(x) = xQ(x^2+1)$. Then $Q((x^2+1)^2+1) = Q(x^2+1)^2 - 1$ or $(x^2+1)Q((x^2+1)^2+1) = x^2Q(x^2+1)^2 + 1$, respectively. Substituting $x^2+1 = y$ yields $Q(y^2+1) = Q(y)^2 + 1$ and $yQ(y^2+1) = (y-1)Q(y)^2 + 1$, respectively.

   Suppose that $yQ(y^2+1) = (y-1)Q(y)^2 + 1$. Setting $y = 1$ we obtain that $Q(2) = 1$. Note that, if $a \neq 0$ and $Q(a) = 1$, then also $aQ(a^2+1) = (a-1) + 1$ and hence $Q(a^2+1) = 1$. We thus obtain an infinite sequence of points at which Q takes value 1, namely the sequence given by $a_0 = 2$ and $a_{n+1} = a_n^2 + 1$. Therefore $Q \equiv 1$.

   It follows that if $Q \not\equiv 1$, then $P(x) = Q(x^2+1)$. Now we can easily list all solutions: these are the polynomials of the form $T(T(\cdots(T(x))\cdots))$, where $T(x) = x^2 + 1$.

7. If a polynomial P with real coefficients satisfies for all x

   $$P(\cos x) = P(\sin x),$$

   prove that there exists a polynomial Q such that for all x, $P(x) = Q(x^4 - x^2)$.

   Solution. It follows from the condition of the problem that $P(-\sin x) = P(\sin x)$, so $P(-t) = P(t)$ for infinitely many t; hence the polynomials $P(x)$ and $P(-x)$ coincide. Therefore $P(x) = S(x^2)$ for some polynomial S. Now $S(\cos^2 x) = S(\sin^2 x)$ for all x, so $S(1-t) = S(t)$ for infinitely many t, which gives us $S(x) \equiv S(1-x)$. This is equivalent to $R(x - \frac{1}{2}) = R(\frac{1}{2} - x)$, i.e. $R(y) \equiv R(-y)$, where R is the polynomial such that $S(x) = R(x - \frac{1}{2})$. Now $R(x) = T(x^2)$ for some polynomial T, and finally, $P(x) = S(x^2) = R(x^2 - \frac{1}{2}) = T(x^4 - x^2 + \frac{1}{4}) = Q(x^4 - x^2)$ for some polynomial Q.

8. Find all quadruples of polynomials (P_1, P_2, P_3, P_4) such that, whenever natural numbers x, y, z, t satisfy $xy - zt = 1$, it holds that $P_1(x)P_2(y) - P_3(z)P_4(t) = 1$.

   Solution. Clearly $P_1(x)P_2(y) = P_2(x)P_1(y)$ for all natural numbers x and y. This implies that $P_2(x)/P_1(x)$ does not depend on x. Hence $P_2 = cP_1$ for some constant c. Analogously, $P_4 = dP_3$ for some constant d. Now we have $cP_1(x)P_1(y) - dP_3(z)P_3(t) = 1$ for all natural x, y, z, t with $xy - zt = 1$. Moreover, we see that $P_1(x)P_1(y)$ depends only on xy, i.e. $f(x) = P_1(x)P_1(n/x)$ is the same for all positive divisors x of natural number n. Since $f(x)$ is a rational function and the number of divisors x of n can be arbitrarily large, it follows that f is constant in x, i.e. a polynomial in n. It is easily verified that this is possible only when $P_1(x) = x^n$ for some n. Similarly, $P_3(x) = x^m$ for some m and $c(xy)^n - d(zt)^m = 1$. Therefore $m = n$ and $c = d = 1$, and finally $m = n = 1$. So, $P_1(x) = P_2(x) = P_3(x) = P_4(x) = x$.

9. Find all polynomials $P(x)$ with real coefficients that satisfy the equality

   $$P(a-b) + P(b-c) + P(c-a) = 2P(a+b+c)$$

   for all triples a, b, c of real numbers such that $ab + bc + ca = 0$. *(IMO 2004.2)*

   Solution. Let $P(x) = a_0 + a_1x + \cdots + a_nx^n$. For every $x \in \mathbb{R}$ the triple $(a,b,c) = (6x, 3x, -2x)$ satisfies the condition $ab + bc + ca = 0$. Then the condition on P gives us $P(3x) + P(5x) + P(-8x) = 2P(7x)$ for all x, implying that for all $i = 0, 1, 2, \ldots, n$ the following equality holds:

   $$\left(3^i + 5^i + (-8)^i - 2 \cdot 7^i\right) a_i = 0.$$

Suppose that $a_i \neq 0$. Then $K(i) = 3^i + 5^i + (-8)^i - 2\cdot 7^i = 0$. But $K(i)$ is negative for i odd and positive for $i = 0$ or $i \geq 6$ even. Only for $i = 2$ and $i = 4$ do we have $K(i) = 0$. It follows that $P(x) = a_2x^2 + a_4x^4$ for some real numbers a_2, a_4. It is easily verified that all such $P(x)$ satisfy the required condition.

10. (a) If a real polynomial $P(x)$ satisfies $P(x) \geq 0$ for all x, show that there exist real polynomials $A(x)$ and $B(x)$ such that $P(x) = A(x)^2 + B(x)^2$.

    (b) If a real polynomial $P(x)$ satisfies $P(x) \geq 0$ for all $x \geq 0$, show that there exist real polynomials $A(x)$ and $B(x)$ such that $P(x) = A(x)^2 + xB(x)^2$.

    Solution. Polynomial $P(x)$ can be written in the form

$$P(x) = (x-a_1)^{\alpha_1}\cdots(x-a_k)^{\alpha_k}\cdot(x^2-b_1x+c_1)\cdots(x^2-b_mx+c_m), \qquad (*)$$

    where a_i, b_j, c_j are real numbers such that a_i are distinct and the polynomials $x^2 - b_ix + c_i$ have no real roots.

    It follows from the condition $P(x) \geq 0$ for all x that all the α_i are even, and from the condition $P(x) \geq 0$ for all $x \geq 0$ that $(\forall i)$ either α_i is even or $a_i < 0$. This ensures that each linear or quadratic factor in $(*)$ can be written in the required form $A^2 + B^2$ and/or $A^2 + xB^2$. The well-known formula $(a^2 + \gamma b^2)(c^2 + \gamma d^2) = (ac + \gamma bd)^2 + \gamma(ad - bc)^2$ now gives a required representation for their product $P(x)$.

11. Prove that if the polynomials P and Q have a real root each and

$$P(1 + x + Q(x)^2) = Q(1 + x + P(x)^2),$$

    then $P \equiv Q$.

    Solution. Note that there exists $x = a$ for which $P(a)^2 = Q(a)^2$. This follows from the fact that, if p and q are the respective real roots of P and Q, then $P(p)^2 - Q(p)^2 \leq 0 \leq P(q)^2 - Q(q)^2$, and moreover $P^2 - Q^2$ is continuous. Now $P(b) = Q(b)$ for $b = 1 + a + P(a)^2$. Taking a to be the largest real number for which $P(a) = Q(a)$ leads to an immediate contradiction.

12. If P and Q are monic polynomials with $P(P(x)) = Q(Q(x))$, prove that $P \equiv Q$.

    Solution. Suppose that $R = P - Q \neq 0$ and that $0 < k \leq n-1$ is the degree of $R(x)$. Then

$$P(P(x)) - Q(Q(x)) = [Q(P(x)) - Q(Q(x))] + R(P(x)).$$

    Putting $Q(x) = x^n + \cdots + a_1x + a_0$ we have $Q(P(x)) - Q(Q(x)) = [P(x)^n - Q(x)^n] + \cdots + a_1[P(x) - Q(x)]$, where all summands but the first have a degree at most $n^2 - n$, while the first summand equals $R(x)\cdot\left(P(x)^{n-1} + P(x)^{n-2}Q(x) + \cdots + Q(x)^{n-1}\right)$ and therefore has the degree $n^2 - n + k$ with the leading coefficient n. Hence the degree of $Q(P(x)) - Q(Q(x))$ is $n^2 - n + k$. The degree of $R(P(x))$ is equal to $kn < n^2 - n + k$, from what we conclude that the degree of the difference $P(P(x)) - Q(Q(x))$ is $n^2 - n + k$, a contradiction.

    In the remaining case when $R \equiv c$ is constant, the condition $P(P(x)) = Q(Q(x))$ gives us $Q(Q(x) + c) = Q(Q(x)) - c$, so the equality $Q(y + c) = Q(y) - c$ holds for infinitely many y, implying $Q(y + c) \equiv Q(y) - c$. But this is only possible for $c = 0$.

13. Assume that there exist complex polynomials P, Q, R such that

$$P^a + Q^b = R^c,$$

    where a, b, c are natural numbers. Show that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$.

    Solution. We use the following auxilliary statement.

Lemma 2. *If A, B and C are pairwise coprime polynomials with $A+B=C$, then the degree of each of them is less than the number of different zeroes of the polynomial ABC.*

Proof. Let

$$A(x)=\prod_{i=1}^{k}(x-p_i)^{a_i}, \quad B(x)=\prod_{i=1}^{l}(x-q_i)^{b_i}, \quad C(x)=\prod_{i=1}^{m}(x-r_i)^{c_i}.$$

Writing the condition $A+B=C$ as $A(x)C(x)^{-1}+B(x)C(x)^{-1}=1$ and differentiating it with respect to x gives us

$$A(x)C(x)^{-1}\left(\sum_{i=1}^{k}\frac{a_i}{x-p_i}-\sum_{i=1}^{m}\frac{c_i}{x-r_i}\right)=-B(x)C(x)^{-1}\left(\sum_{i=1}^{l}\frac{b_i}{x-q_i}-\sum_{i=1}^{m}\frac{c_i}{x-r_i}\right),$$

from which we see that $A(x)/B(x)$ can be written as a quotient of two polynomials od degrees not exceeding $k+l+m-1$. Our statement now follows from the fact that A and B are coprime. Apply this statement on polynomials P^a, Q^b, R^c. Each of their degrees $a\deg P$, $b\deg Q$, $c\deg R$ is less than $\deg P+\deg Q+\deg R$ and hence $\frac{1}{a}>\frac{\deg P}{\deg P+\deg Q+\deg R}$, etc. Summing up yields the desired inequality.

Corollary. “The Last Fermat’s theorem” for polynomials.

14. The lateral surface of a cylinder is divided by $n-1$ planes parallel to the base and m meridians into mn cells ($n\geq 1, m\geq 3$). Two cells are called neighbors if they have a common side. Prove that it is possible to write real numbers in the cells, not all zero, so that the number in each cell equals the sum of the numbers in the neighboring cells, if and only if there exist k,l with $n+1\nmid k$ such that $\cos\dfrac{2l\pi}{m}+\cos\dfrac{k\pi}{n+1}=\dfrac{1}{2}$.

Solution. Denote by a_{ij} the number in the intersection of i-th parallel and j-th meridian. We assign to the i-th parallel the polynomial $p_i(x)=a_{i1}+a_{i2}x+\dots+a_{im}x^{m-1}$ and define $p_0(x)=p_{n+1}(x)=0$. The property that each number equals the sum of its neighbors can be written as $p_i(x)=p_{i-1}(x)+p_{i+1}(x)+(x^{m-1}+x)p_i(x)$ modulo x^m-1, i.e.

$$p_{i+1}(x)=(1-x-x^{m-1})p_i(x)-p_{i-1}(x) \pmod{x^m-1}.$$

This sequence of polynomials is entirely determined by term $p_1(x)$. The numbers a_{ij} can be written in the required way if and only if a polynomial $p_1(x)\neq 0$ of degree less than m can be chosen so that $p_{n+1}(x)=0$.

Consider the sequence of polynomials $r_i(x)$ given by $r_0=0$, $r_1=1$ and $r_{i+1}=(1-x-x^{m-1})r_i-r_{i-1}$. Clearly, $p_{n+1}(x)\equiv r_{n+1}(x)p_1(x) \pmod{x^m-1}$. Polynomial $p_1\neq 0$ of degree $<m$ for which $p_{n+1}=0$ exists if and only if $r_{n+1}(x)$ and x^m-1 are not coprime, i.e. if and only if there exists ε such that $\varepsilon^m=1$ and $r_{n+1}(\varepsilon)=0$. Now consider the sequence (x_i) given by $x_0=0, x_1=1$ and $x_{i+1}=(1-\varepsilon-\varepsilon^{m-1})x_i-x_{i-1}$. Let us write $c=1-\varepsilon-\varepsilon^{m-1}$ and denote by u_1,u_2 the zeroes of polynomial x^2-cx+1. The general term of the above recurrent sequence is $x_i=\dfrac{u_1^i-u_2^i}{u_1-u_2}$ if $u_1\neq u_2$ and $x_i=iu_1^i$ if $u_1=u_2$. The latter case is clearly impossible. In the former case ($u_1\neq u_2$) equality $x_{n+1}=0$ is equivalent to $u_1^{n+1}=u_2^{n+1}$ and hence to $\omega^{n+1}=1$, where $u_1=u_2\omega$, which holds if and only if $(\exists u_2)\ u_2^2\omega=1$ and $u_2(1+\omega)=c$. Therefore $(1+\omega)^2=c^2\omega$, so

$$2+\omega+\overline{\omega}=(1-\varepsilon-\overline{\varepsilon})^2.$$

Now if $\omega=\cos\frac{2k\pi}{n+1}+i\sin\frac{2k\pi}{n+1}$ and $\varepsilon=\cos\frac{2l\pi}{m}+i\sin\frac{2l\pi}{m}$, the above equality becomes the desired one.

Functional Equations

Marko Radovanović
radmarko@yahoo.com

Contents

1 Basic Methods For Solving Functional Equations

- Substituting the values for variables. The most common first attempt is with some constants (eg. 0 or 1), after that (if possible) some expressions which will make some part of the equation to become constant. For example if $f(x+y)$ appears in the equations and if we have found $f(0)$ then we plug $y=-x$. Substitutions become less obvious as the difficulty of the problems increase.

- Mathematical induction. This method relies on using the value $f(1)$ to find all $f(n)$ for n integer. After that we find $f\left(\frac{1}{n}\right)$ and $f(r)$ for rational r. This method is used in problems where the function is defined on $\mathbb{Q}$ and is very useful, especially with easier problems.

- Investigating for injectivity or surjectivity of functions involved in the equaiton. In many of the problems these facts are not difficult to establish but can be of great importance.

- Finding the fixed points or zeroes of functions. The number of problems using this method is considerably smaller than the number of problems using some of the previous three methods. This method is mostly encountered in more difficult problems.

- Using the Cauchy's equation and equation of its type.

- Investigating the monotonicity and continuity of a function. Continuity is usually given as additional condition and as the monotonicity it usually serves for reducing the problem to Cauchy's equation. If this is not the case, the problem is on the other side of difficulty line.

- Assuming that the function at some point is greater or smaller then the value of the function for which we want to prove that is the solution. Most often it is used as continuation of the method of mathematical induction and in the problems in which the range is bounded from either side.

- Making recurrent relations. This method is usually used with the equations in which the range is bounded and in the case when we are able to find a relashionship between $f(f(n))$, $f(n)$, and n.

- Analyzing the set of values for which the function is equal to the assumed solution. The goal is to prove that the described set is precisely the domain of the function.
- Substituting the function. This method is often used to simplify the given equation and is seldom of crucial importance.
- Expressing functions as sums of odd and even. Namely each function can be represented as a sum of one even and one odd function and this can be very handy in treating ”linear” functional equations involving many functions.
- Treating numbers in a system with basis different than 10. Of course, this can be used only if the domain is $\mathbb{N}$.
- For the end let us emphasize that it is very important to guess the solution at the beginning. This can help a lot in finding the appropriate substitutions. Also, at the end of the solution, DON'T FORGET to verify that your solution satisfies the given condition.

2 Cauchy Equation and Equations of the Cauchy type

The equation $f(x+y) = f(x) + f(y)$ is called the Cauchy equation. If its domain is $\mathbb{Q}$, it is well-known that the solution is given by $f(x) = xf(1)$. That fact is easy to prove using mathematical induction. The next problem is simply the extention of the domain from $\mathbb{Q}$ to $\mathbb{R}$. With a relatively easy counter-example we can show that the solution to the Cauchy equation in this case doesn't have to be $f(x) = xf(1)$. However there are many additional assumptions that forces the general solution to be of the described form. Namely if a function f satisfies any of the conditions:

- monotonicity on some interval of the real line;
- continuity;
- boundedness on some interval;
- positivity on the ray $x \geq 0$;

then the general solution to the Cauchy equation $f : \mathbb{R} \to S$ has to be $f(x) = xf(1)$.

The following equations can be easily reduced to the Cauchy equation.

- All continuous functions $f : \mathbb{R} \to (0,+\infty)$ satisfying $f(x+y) = f(x)f(y)$ are of the form $f(x) = a^x$. Namely the function $g(x) = \log f(x)$ is continuous and satisfies the Cauchy equation.
- All continuous functions $f : (0,+\infty) \to \mathbb{R}$ satisfying $f(xy) = f(x) + f(y)$ are of the form $f(x) = \log_a x$. Now the function $g(x) = f(a^x)$ is continuous and satisfies the Cauchy equation.
- All continuous functions $f : (0,+\infty) \to (0,+\infty)$ satisfying $f(xy) = f(x)f(y)$ are $f(x) = x^t$, where $t = \log_a b$ and $f(a) = b$. Indeed the function $g(x) = \log f(a^x)$ is continuous and satisfies the Cauchy equation.

3 Problems with Solutions

The following examples should illustrate the previously outlined methods.

Problem 1. *Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(1) = 2$ and $f(xy) = f(x)f(y) - f(x+y) + 1$.*

Solution. This is a classical example of a problem that can be solved using mathematical induction. Notice that if we set $x=1$ and $y=n$ in the original equation we get $f(n+1)=f(n)+1$, and since $f(1)=2$ we have $f(n)=n+1$ for every natural number n. Similarly for $x=0$ and $y=n$ we get $f(0)n=f(n)-1=n$, i.e. $f(0)$. Now our goal is to find $f(z)$ for each $z\in\mathbb{Z}$. Substituting $x=-1$ and $y=1$ in the original equation gives us $f(-1)=0$, and setting $x=-1$ and $y=n$ gives $f(-n)=-f(n-1)+1=-n+1$. Hence $f(z)=z+1$ for each $z\in\mathbb{Z}$. Now we have to determine $f\left(\frac{1}{n}\right)$. Plugging $x=n$ and $y=\frac{1}{n}$ we get

$$f(1)=(n+1)f\left(\frac{1}{n}\right)-f\left(n+\frac{1}{n}\right)+1. \tag{1}$$

Furthermore for $x=1$ and $y=m+\frac{1}{n}$ we get $f\left(m+1+\frac{1}{n}\right)=f\left(m+\frac{1}{n}\right)+1$, hence by the mathematical induction $f\left(m+\frac{1}{n}\right)=m+f\left(\frac{1}{n}\right)$. Iz (1) we now have

$$f\left(\frac{1}{n}\right)=\frac{1}{n}+1,$$

for every natural number n. Furthermore for $x=m$ and $y=\frac{1}{n}$ we get $f\left(\frac{m}{n}\right)=\frac{m}{n}+1$, i.e. $f(r)=r+1$, for every positive rational number r. Setting $x=-1$ and $y=r$ we get $f(-r)=-f(r-1)+1=-r+1$ as well hence $f(x)=x+1$, for each $x\in\mathbb{Q}$.
Verification: Since $xy+1=(x+1)(y+1)-(x+y+1)+1$, for all $x,y\in\mathbb{Q}$, f is the solution to our equation. △

Problem 2. *(Belarus 1997) Find all functions $g:\mathbb{R}\to\mathbb{R}$ such that for arbitrary real numbers x and y:*

$$g(x+y)+g(x)g(y)=g(xy)+g(x)+g(y).$$

Solution. Notice that $g(x)=0$ and $g(x)=2$ are obviously solutions to the given equation. Using mathematical induction it is not difficult to prove that if g is not equal to one of these two functions then $g(x)=x$ for all $x\in\mathbb{Q}$. It is also easy to prove that $g(r+x)=r+g(x)$ and $g(rx)=rg(x)$, where r is rational and x real number. Particularly from the second equation for $r=-1$ we get $g(-x)=-g(x)$, hence setting $y=-x$ in the initial equation gives $g(x)^2=g(x^2)$. This means that $g(x)\geq 0$ for $x\geq 0$. Now we use the standard method of extending to $\mathbb{R}$. Assume that $g(x)<x$. Choose $r\in\mathbb{Q}$ such that $g(x)<r<x$. Then

$$r>g(x)=g(x-r)+r\geq r,$$

which is clearly a contradiction. Similarly from $g(x)>x$ we get another contradiction. Thus we must have $g(x)=x$ for every $x\in\mathbb{R}$. It is easy to verify that all three functions satisfy the given functional equation. △

Problem 3. *The function $f:\mathbb{R}\to\mathbb{R}$ satisfies $x+f(x)=f(f(x))$ for every $x\in\mathbb{R}$. Find all solutions of the equation $f(f(x))=0$.*

Solution. The domain of this function is $\mathbb{R}$, so there isn't much hope that this can be solved using mathematical induction. Notice that $f(f(x))-f(x)=x$ and if $f(x)=f(y)$ then clearly $x=y$. This means that the function is injective. Since $f(f(0))=f(0)+0=f(0)$, because of injectivity we must have $f(0)=0$, implying $f(f(0))=0$. If there were another x such that $f(f(x))=0=f(f(0))$, injectivity would imply $f(x)=f(0)$ and $x=0$. △

Problem 4. *Find all injective functions $f:\mathbb{N}\to\mathbb{R}$ that satisfy:*

$$(a)\ f(f(m)+f(n))=f(f(m))+f(n),\quad (b)\ f(1)=2,\ f(2)=4.$$

Solution. Setting $m = 1$ and n first, and $m = n$, $n = 1$ afterwards we get

$$f(f(1)+f(n)) = f(f(1))+f(n), \quad f(f(n)+f(1)) = f(f(n))+f(1).$$

Let us emphasize that this is one standard idea if the expression on one side is symmetric with respect to the variables while the expression on the other side is not. Now we have $f(f(n)) = f(n)-f(1)+f(f(1)) = f(n)-2+f(2) = f(n)+2$. From here we conclude that $f(n) = m$ implies $f(m) = m+2$ and now the induction gives $f(m+2k) = m+2k+2$, for every $k \geq 0$. Specially if $f(1) = 2$ then $f(2n) = 2n+2$ for all positive integers n. The injectivity of f gives that at odd numbers (except 1) the function has to take odd values. Let p be the smallest natural number such that for some k $f(k) = 2p+1$. We have $f(2p+2s+1) = 2p+2s+3$ for $s \geq 0$. Therefore the numbers $3,5,\dots,2p-1$ are mapped into $1,3,\dots,2p+1$. If $f(t) = 1$ for some t, then for $m = n = t$ $4 = f(2) = f(f(t)+f(t)) = f(f(t))+f(t) = 3$, which is a contradiction. If for some t such that $f(t) = 3$ then $f(3+2k) = 5+2k$, which is a contradiction to the existence of such t. It follows that the numbers $3,5,\dots,2p-1$ are mapped into $5,7,\dots,2p+1$. Hence $f(3+2k) = 5+2k$. Thus the solution is $f(1) = 2$ and $f(n) = n+2$, for $n \geq 2$.
It is easy to verify that the function satisfies the given conditions. △

Problem 5. *(BMO 1997, 2000) Solve the functional equation*

$$f(xf(x)+f(y)) = y+f(x)^2, \ x,y \in \mathbb{R}.$$

Solution. In probelms of this type it is usually easy to prove that the functions are injective or surjective, if the functions are injective/surjective. In this case for $x = 0$ we get $f(f(y)) = y+f(0)^2$. Since the function on the right-hand side is surjective the same must hold for the function on the left-hand side. This implies the surjectivity of f. Injectivity is also easy to establish. Now there exists t such that $f(t) = 0$ and substitution $x = 0$ and $y = t$ yields $f(0) = t+f(0)^2$. For $x = t$ we get $f(f(y)) = y$. Therefore $t = f(f(t)) = f(0) = t+f(0)^2$, i.e. $f(0) = 0$. Replacing x with $f(x)$ gives

$$f(f(x)x+f(y)) = x^2+y,$$

hence $f(x)^2 = x^2$ for every real number x. Consider now the two cases:
First case $f(1) = 1$. Plugging $x = 1$ gives $f(1+f(y)) = 1+y$, and after taking squares $(1+y)^2 = f(1+f(y))^2 = (1+f(y))^2 = 1+2f(y)+f(y)^2 = 1+2f(y)+y^2$. Clearly in this case we have $f(y) = y$ for every real y.
Second case $f(1) = -1$. Plugging $x = -1$ gives $f(-1+f(y)) = 1+y$, and after taking squares $(1+y)^2 = f(-1+f(y))^2 = (-1+f(y))^2 = 1-2f(y)+f(y)^2 = 1-2f(y)+y^2$. Now we conclude $f(y) = -y$ for every real number y.
It is easy to verify that $f(x) = x$ and $f(x) = -x$ are indeed the solutions. △

Problem 6. *(IMO 1979, shortlist) Given a function $f : \mathbb{R} \to \mathbb{R}$, if for every two real numbers x and y the equality $f(xy+x+y) = f(xy)+f(x)+f(y)$ holds, prove that $f(x+y) = f(x)+f(y)$ for every two real numbers x and y.*

Solution. This is a clasical example of the equation that solution is based on a careful choice of values that are plugged in a functional equation. Plugging in $x = y = 0$ we get $f(0) = 0$. Plugging in $y = -1$ we get $f(x) = -f(-x)$. Plugging in $y = 1$ we get $f(2x+1) = 2f(x)+f(1)$ and hence $f(2(u+v+uv)+1) = 2f(u+v+uv)+f(1) = 2f(uv)+2f(u)+2f(v)+f(1)$ for all real u and v. On the other hand, plugging in $x = u$ and $y = 2v+1$ we get $f(2(u+v+uv)+1) = f(u+(2v+1)+u(2v+1)) = f(u)+2f(v)+f(1)+f(2uv+u)$. Hence it follows that $2f(uv)+2f(u)+2f(v)+f(1) = f(u)+2f(v)+f(1)+f(2uv+u)$, i.e.,

$$f(2uv+u) = 2f(uv)+f(u). \qquad (1)$$

Plugging in $v = -1/2$ we get $0 = 2f(-u/2)+f(u) = -2f(u/2)+f(u)$. Hence, $f(u) = 2f(u/2)$ and consequently $f(2x) = 2f(x)$ for all reals. Now (1) reduces to $f(2uv+u) = f(2uv)+f(u)$. Plugging in $u = y$ and $x = 2uv$, we obtain $f(x)+f(y) = f(x+y)$ for all nonzero reals x and y. Since $f(0) = 0$, it trivially holds that $f(x+y) = f(x)+f(y)$ when one of x and y is 0. △

Problem 7. *Does there exist a function $f:\mathbb{R}\to\mathbb{R}$ such that $f(f(x))=x^2-2$ for every real number x?*

Solution. After some attempts we can see that none of the first three methods leads to a progress. Notice that the function g of the right-hand side has exactly 2 fixed points and that the function $g\circ g$ has exactly 4 fixed points. Now we will prove that there is no function f such that $f\circ f=g$. Assume the contrary. Let a,b be the fixed points of g, and a,b,c,d the fixed points of $g\circ g$. Assume that $g(c)=y$. Then $c=g(g(c))=g(y)$, hence $g(g(y))=g(c)=y$ and y has to be on of the fixed points of $g\circ g$. If $y=a$ then from $a=g(a)=g(y)=c$ we get a contradiction. Similarly $y\neq b$, and since $y\neq c$ we get $y=d$. Thus $g(c)=d$ and $g(d)=c$. Furthermore we have $g(f(x))=f(f(f(x)))=f(g(x))$. Let $x_0\in\{a,b\}$. We immediately have $f(x_0)=f(g(x_0))=g(f(x_0))$, hence $f(x_0)\in\{a,b\}$. Similarly if $x_1\in\{a,b,c,d\}$ we get $f(x_1)\in\{a,b,c,d\}$, and now we will prove that this is not possible. Take first $f(c)=a$. Then $f(a)=f(f(c))=g(c)=d$ which is clearly impossible. Similarly $f(c)\neq b$ and $f(c)\neq c$ (for otherwise $g(c)=c$) hence $f(c)=d$. However we then have $f(d)=f(f(c))=g(c)=d$, which is a contradiction, again. This proves that the required f doesn't exist. △

Problem 8. *Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that $f(x)f(yf(x))=f(x+y)$ for every two positive real numbers x,y.*

Solution. Obviously $f(x)\equiv 1$ is one solution to the problem. The idea is to find y such that $yf(x)=x+y$ and use this to determine $f(x)$. For every x such that $\dfrac{x}{f(x)-1}\geq 0$ we can find such y and from the given condition we get $f(x)=1$. However this is a contradiction since we got that $f(x)>1$ implies $f(x)=1$. One of the consequences is that $f(x)\leq 1$. Assume that $f(x)<1$ for some x. From the given equation we conclude that f is non-increasing (because $f(yf(x))\leq 1$). Let us prove that f is decreasing. In order to do that it is enough to prove that $f(x)<1$, for each x. Assume that $f(x)=1$ for every $x\in(0,a)$ $(a>0)$. Substituting $x=y=\dfrac{2a}{3}$ in the given equation we get the obvious contradiction. This means that the function is decreasing and hence it is injective. Again everything will revolve around the idea of getting rid of $f(yf(x))$. Notice that $x+y>yf(x)$, therefore

$$f(x)f(yf(x))=f(x+y)=f(yf(x)+x+y-yf(x))=f(yf(x))f\Big(f\Big(yf(x)\Big)(x+y-yf(x))\Big),$$

i.e. $f(x)=f\Big(f\Big(yf(x)\Big)(x+y-yf(x))\Big)$. The injectivity of f implies that $x=f\Big(yf(x)\Big)(x+y-yf(x))$. If we plug $f(x)=a$ we get

$$f(y)=\frac{1}{1+\alpha z},$$

where $\alpha=\dfrac{1-f(a)}{af(a)}$, and according to our assumption $\alpha>0$.

It is easy to verify that $f(x)=\dfrac{1}{1+\alpha x}$, for $\alpha\in\mathbb{R}^+$, and $f(x)\equiv 1$ satisfy the equation. △

Problem 9. *(IMO 2000, shortlist) Find all pairs of functions $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ such that for every two real numbers x,y the following relation holds:*

$$f(x+g(y))=xf(y)-yf(x)+g(x).$$

Solution. Let us first solve the problem under the assumption that $g(\alpha)=0$ for some α.

Setting $y=\alpha$ in the given equation yields $g(x)=(\alpha+1)f(x)-xf(\alpha)$. Then the given equation becomes $f(x+g(y))=(\alpha+1-y)f(x)+(f(y)-f(\alpha))x$, so setting $y=\alpha+1$ we get $f(x+n)=mx$, where $n=g(\alpha+1)$ and $m=f(\alpha+1)-f(\alpha)$. Hence f is a linear function, and consequently g is also linear. If we now substitute $f(x)=ax+b$ and $g(x)=cx+d$ in the given equation and compare the coefficients, we easily find that

$$f(x)=\frac{cx-c^2}{1+c}\quad\text{and}\quad g(x)=cx-c^2,\qquad c\in\mathbb{R}\setminus\{-1\}.$$

Now we prove the existence of α such that $g(\alpha)=0$. If $f(0)=0$ then putting $y=0$ in the given equation we obtain $f(x+g(0))=g(x)$, so we can take $\alpha=-g(0)$.

Now assume that $f(0)=b\neq 0$. By replacing x by $g(x)$ in the given equation we obtain $f(g(x)+g(y))=g(x)f(y)-yf(g(x))+g(g(x))$ and, analogously, $f(g(x)+g(y))=g(y)f(x)-xf(g(y))+g(g(y))$. The given functional equation for $x=0$ gives $f(g(y))=a-by$, where $a=g(0)$. In particular, g is injective and f is surjective, so there exists $c\in\mathbb{R}$ such that $f(c)=0$. Now the above two relations yield

$$g(x)f(y)-ay+g(g(x))=g(y)f(x)-ax+g(g(y)). \qquad (1)$$

Plugging $y=c$ in (1) we get $g(g(x))=g(c)f(x)-ax+g(g(c))+ac=kf(x)-ax+d$. Now (1) becomes $g(x)f(y)+kf(x)=g(y)f(x)+kf(y)$. For $y=0$ we have $g(x)b+kf(x)=af(x)+kb$, whence

$$g(x)=\frac{a-k}{b}f(x)+k.$$

Note that $g(0)=a\neq k=g(c)$, since g is injective. From the surjectivity of f it follows that g is surjective as well, so it takes the value 0. △

Problem 10. *(IMO 1992, shortlist) Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ which satisfy*

$$f(f(x))+af(x)=b(a+b)x.$$

Solution. This is a typical example of a problem that is solved using recurrent equations. Let us define x_n inductively as $x_n=f(x_{n-1})$, where $x_0\geq 0$ is a fixed real number. It follows from the given equation in f that $x_{n+2}=-ax_{n+1}+b(a+b)x_n$. The general solution to this equation is of the form

$$x_n=\lambda_1 b^n+\lambda_2(-a-b)^n,$$

where $\lambda_1,\lambda_2\in\mathbb{R}$ satisfy $x_0=\lambda_1+\lambda_2$ and $x_1=\lambda_1 b-\lambda_2(a+b)$. In order to have $x_n\geq 0$ for all n we must have $\lambda_2=0$. Hence $x_0=\lambda_1$ and $f(x_0)=x_1=\lambda_1 b=bx_0$. Since x_0 was arbitrary, we conclude that $f(x)=bx$ is the only possible solution of the functional equation. It is easily verified that this is indeed a solution. △

Problem 11. *(Vietnam 2003) Let F be the set of all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ which satisfy the inequality $f(3x)\geq f(f(2x))+x$, for every positive real number x. Find the largest real number α such that for all functions $f\in F$: $f(x)\geq\alpha\cdot x$.*

Solution. We clearly have that $\frac{x}{2}\in F$, hence $\alpha\leq\frac{1}{2}$. Furthermore for every function $f\in F$ we have $f(x)\geq\frac{x}{3}$. The idea is the following: Denote $\frac{1}{3}=\alpha_1$ and form a sequence $\{\alpha_n\}$ for which $f(x)\geq\alpha_n x$ and which will (hopefully) tend to $\frac{1}{2}$. This would imply that $\alpha\geq\frac{1}{2}$, and hence $\alpha=\frac{1}{2}$. Let us constract a recurrent relation for α_k. Assume that $f(x)\geq\alpha_k x$, for every $x\in\mathbb{R}^+$. From the given inequality we have

$$f(3x)\geq f(f(2x))+x\geq\alpha_k f(2x)+x\geq\alpha_k\cdot\alpha_k\cdot 2x+x=\alpha_{k+1}\cdot 3x.$$

This means that $\alpha_{n+1}=\frac{2\alpha_n^2+1}{3}$. Let us prove that $\lim_{n\to+\infty}\alpha_n=\frac{1}{2}$. This is a standard problem. It is easy to prove that the sequence α_k is increasing and bounded above by $\frac{1}{2}$. Hence it converges and its limit α satisfies $\alpha=\frac{2\alpha^2+1}{3}$, i.e. $\alpha=\frac{1}{2}$ (since $\alpha<1$). △

Problem 12. *Find all functions $f,g,h:\mathbb{R}\to\mathbb{R}$ that satisfy*

$$f(x+y)+g(x-y)=2h(x)+2h(y).$$

Solution. Our first goal is to express f and g using h and get the equation involving h only. First taking $y=x$ and substituting $g(0)=a$ we get $f(2x)=4h(x)-a$. Furthermore by putting $y=0$ we get $g(x)=2h(x)+2b-4h\left(\frac{x}{2}\right)+a$, where $h(0)=b$. Now the original equation can be written as

$$2\left[h\left(\frac{x+y}{2}\right)+h\left(\frac{x-y}{2}\right)\right]+h(x-y)+b=h(x)+h(y). \tag{2}$$

Let $H(x)=h(x)-b$. These "longer" linear expressions can be easily handled if we express functions in form of the sum of an even and odd function, i.e. $H(x)=H_e(x)+H_o(x)$. Substituting this into (2) and writing the same expressions for $(-x,y)$ and $(x,-y)$ we can add them together and get:

$$2\left[H_e\left(\frac{x-y}{2}\right)+H_e\left(\frac{x+y}{2}\right)\right]+H_e(x-y)=H_e(x)+H_e(y). \tag{3}$$

If we set $-y$ in this expression and add to (3) we get (using $H_e(y)=H_e(-y)$)

$$H_e(x+y)-H_e(x-y)=2H_e(x)+2H_e(y).$$

The last equation is not very difficult. Mathematical induction yields $H_e(r)=\alpha r^2$, for every rational number r. From the continuity we get $H_e(x)=\alpha x^2$. Similar method gives the simple relation for H_o

$$H_o(x+y)+H_o(x-y)=2H_o(x).$$

This is a Cauchy equation hence $H_o(x)=\beta x$. Thus $h(x)=\alpha x^2+\beta x+b$ and substituting for f and g we get:

$$f(x)=\alpha x^2+2\beta x+4b-a, \quad g(x)=\alpha x^2+a.$$

It is easy to verify that these functions satisfy the given conditions.

Problem 13. *Find all functions $f:\mathbb{Q}\to\mathbb{Q}$ for which*

$$f(xy)=f(x)f(y)-f(x+y)+1.$$

Solve the same problem for the case $f:\mathbb{R}\to\mathbb{R}$.

Solution. It is not hard to see that for $x=y=0$ we get $(f(0)-1)^2=0$, i.e. $f(0)=1$. Furthermore, setting $x=1$ and $y=-1$ gives $f(-1)=f(1)f(-1)$, hence $f(-1)=0$ or $f(1)=1$. We will separate this into two cases:

1° Let $f(-1)=0$. In this innocent-looking problems that are resistent to usual ideas it is sometimes successful to increase the number of variables, i.e. to set yz instead of y:

$$f(xyz)=f(x)f(yz)-f(x+yz)+1=f(x)(f(y)f(z)-f(y+z)+1)-f(x+yz)+1.$$

Although it seems that the situation is worse and running out of control, that is not the case. Namely the expression on the left-hand side is symmetric, while the one on the right-hand side is not. Writing the same expression for x and equating gives

$$f(x)f(y+z)-f(x)+f(x+yz)=f(z)f(x+y)-f(z)+f(xy+z). \tag{4}$$

Setting $z=-1$ (we couldn't do that at the beginning, since $z=1$ was fixed) we get $f(x)f(z-1)-f(x)+f(x-y)=f(xy-1)$, and setting $x=1$ in this equality gives

$$f(y-1)(1-f(1))=f(1-y)-f(1). \tag{5}$$

Setting $y=2$ gives $f(1)(2-f(1))=0$, i.e. $f(1)=0$ or $f(1)=2$. This means that we have two cases here as well:

1.1° If $f(1)=0$, then from (5) plugging $y+1$ instead of y we get $f(y)=f(-y)$. Setting $-y$ instead of y in the initial equality gives $f(xy)=f(x)f(y)-f(x-y)+1$, hence $f(x+y)=f(x-y)$, for every two rational numbers x and y. Specially for $x=y$ we get $f(2x)=f(0)=1$, for all $x\in\mathbb{Q}$. However this is a contradiction with $f(1)=0$. In this case we don't have a solution.

1.2° If $f(1)=2$, setting $y+1$ instead of y in (5) gives $1-f(y)=f(-y)-1$. It is clear that we should do the substitution $g(x)=1-f(x)$ because the previous equality gives $g(-x)=-g(x)$, i.e. g is odd. Furthermore substituting g into the original equality gives

$$g(xy)=g(x)+g(y)-g(x)g(y)-g(x+y). \tag{6}$$

Setting $-y$ instead of y we get $-g(xy)=g(x)-g(y)+g(x)g(y)-g(x-y)$, and adding with (6) yields $g(x+y)+g(x-y)=2g(x)$. For $x=y$ we have $g(2x)=2g(x)$ therefore we get $g(x+y)+g(x-y)=g(2x)$. This is a the Cauchy equation and since the domain is $\mathbb{Q}$ we get $g(x)=rx$ for some rational number r. Plugging this back to (6) we obtain $r=-1$, and easy verification shows that $f(x)=1+x$ satisfies the conditions of the problem.

2° Let $f(1)=1$. Setting $z=1$ in (4) we get

$$f(xy+1)-f(x)f(y+1)+f(x)=1,$$

hence for $y=-1$ we get $f(1-x)=1$, for every rational x. This means that $f(x)\equiv 1$ and this function satisfies the given equation.

Now let us solve the problem where $f:\mathbb{R}\to\mathbb{R}$. Notice that we haven't used that the range is $\mathbb{Q}$, hence we conclude that for all rational numbers q $f(q)=q+1$, or $f(q)\equiv 1$. If $f(q)=1$ for all rational numbers q, it can be easily shown that $f(x)\equiv 1$. Assume that $f(q)\not\equiv 1$. From the above we have that $g(x)+g(y)=g(x+y)$, hence it is enough to prove monotonicity. Substitute $x=y$ in (6) and use $g(2x)=2g(x)$ to get $g(x^2)=-g(x)^2$. Therefore for every positive r the value $g(r)$ is non-positive. Hence if $y>x$, i.e. $y=x+r^2$ we have $g(y)=g(x)+g(r^2)\le g(x)$, and the function is decreasing. This means that $f(x)=1+\alpha x$ and after some calculation we get $f(x)=1+x$. It is easy to verify that so obtained functions satisfy the given functional equation. △

Problem 14. *(IMO 2003, shortlist) Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ that satisfy the following conditions:*

(i) $f(xyz)+f(x)+f(y)+f(z)=f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$

(ii) $f(x)<f(y)$ *for all* $1\le x<y$.

Solution. First notice that the solution of this functional equation is not one of the common solutions that we are used to work with. Namely one of the solutions is $f(x)=x+\dfrac{1}{x}$ which tells us that this equality is unlikely to be shown reducing to the Cauchy equation. First, setting $x=y=z=1$ we get $f(1)=2$ (since $f(1)>0$). One of the properties of the solution suggested above is $f(x)=f(1/x)$, and proving this equality will be our next step. Putting $x=ts$, $y=\frac{t}{s}$, $z=\frac{s}{t}$ in (i) gives

$$f(t)f(s)=f(ts)+f(t/s). \tag{7}$$

In particular, for $s=1$ the last equality yields $f(t)=f(1/t)$; hence $f(t)\ge f(1)=2$ for each t. It follows that there exists $g(t)\ge 1$ such that $f(t)=g(t)+\frac{1}{g(t)}$. Now it follows by induction from (7) that $g(t^n)=g(t)^n$ for every integer n, and therefore $g(t^q)=g(t)^q$ for every rational q. Consequently, if $t>1$ is fixed, we have $f(t^q)=a^q+a^{-q}$, where $a=g(t)$. But since the set of a^q ($q\in\mathbb{Q}$) is dense in $\mathbb{R}^+$ and f is monotone on $(0,1]$ and $[1,\infty)$, it follows that $f(t^r)=a^r+a^{-r}$ for every real r. Therefore, if k is such that $t^k=a$, we have

$$f(x)=x^k+x^{-k} \qquad \text{for every } x\in\mathbb{R}. \ \triangle$$

Problem 15. *Find all functions $f : [1,\infty) \to [1,\infty)$ that satisfy:*

(i) $f(x) \le 2(1+x)$ *for every* $x \in [1,\infty)$*;*

(ii) $xf(x+1) = f(x)^2 - 1$ *for every* $x \in [1,\infty)$.

Solution. It is not hard to see that $f(x) = x+1$ is a solution. Let us prove that this is the only solution. Using the given conditions we get

$$f(x)^2 = xf(x+1)+1 \le x(2(x+1))+1 < 2(1+x)^2,$$

i.e. $f(x) \le \sqrt{2}(1+x)$. With this we have found the upper bound for $f(x)$. Since our goal is to prove $f(x) = x+1$ we will use the same method for lowering the upper bound. Similarly we get

$$f(x)^2 = xf(x+1)+1 \le x(\sqrt{2}(x+1))+1 < 2^{1/4}(1+x)^2.$$

Now it is clear that we should use induction to prove

$$f(x) < 2^{1/2^k}(1+x),$$

for every k. However this is shown in the same way as the previous two inequalities. Since $2^{1/2^k} \to 1$ as $k \to +\infty$, hence for fixed x we can't have $f(x) > x+1$. This implies $f(x) \le x+1$ for every real number $x \ge 1$. It remains to show that $f(x) \ge x+1$, for $x \ge 1$. We will use the similar argument. From the fact that the range is $[1,+\infty)$ we get $\frac{f(x)^2-1}{x} = f(x+1) \ge 1$, i.e. $f(x) \ge \sqrt{x+1} > x^{1/2}$. We further have $f(x)^2 = 1+xf(x+1) > 1+x\sqrt{x+2} > x^{3/2}$ and similarly by induction

$$f(x) > x^{1-1/2^k}.$$

Passing to the limit we further have $f(x) \ge x$. Now again from the given equality we get $f(x)^2 = 1+xf(x+1) \ge (x+1/2)^2$, i.el $f(x) \ge x+1/2$. Using the induction we get $f(x) \ge x+1-\frac{1}{2^k}$, and passing to the limit we get the required inequality $f(x) \ge x+1$. △

Problem 16. *(IMO 1999, probelm 6) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x-f(y)) = f(f(y)) + xf(y) + f(x) - 1.$$

Solution. Let $A = \{f(x)\,|\,x \in \mathbb{R}\}$, i.e. $A = f(\mathbb{R})$. We will determine the value of the function on A. Let $x = f(y) \in A$, for some y. From the given equality we have $f(0) = f(x)+x^2+f(x)-1$, i.e.

$$f(x) = \frac{c+1}{2} - \frac{x^2}{2},$$

where $f(0) = c$. Now it is clear that we have to analyze set A further. Setting $x = y = 0$ in the original equation we get $f(-c) = f(c)+c-1$, hence $c \ne 0$. Furthermore, plugging $y = 0$ in the original equation we get $f(x-c)-f(x) = cx+f(c)-1$. Since the range of the function (on x) on the right-hand side is entire $\mathbb{R}$, we get $\{f(x-c)-f(x)\,|\,x \in \mathbb{R}\} = \mathbb{R}$, i.e. $A-A = \mathbb{R}$. Hence for every real number x there are real numbers $y_1, y_2 \in A$ such that $x = y_1 - y_2$. Now we have

$$\begin{aligned} f(x) &= f(y_1-y_2) = f(y_1-f(z)) = f(f(z)) + y_1 f(z) + f(y_1) - 1 \\ &= f(y_1)+f(y_2)+y_1y_2-1 = c - \frac{x^2}{2}. \end{aligned}$$

From the original equation we easily get $c = 1$. It is easy to show that the function $f(x) = 1 - \frac{x^2}{2}$ satisfies the given equation. △

Problem 17. *Given an integer n, let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying $f(0)=0$, $f(1)=1$, and $f^{(n)}(x)=x$, for every $x\in[0,1]$. Prove that $f(x)=x$ for each $x\in[0,1]$.*

Solution. First from $f(x)=f(y)$ we have $f^{(n)}(x)=f^{(n)}(y)$, hence f is injective. The idea for what follows is clear once we look at the graphical representation. Namely from the picture it can be easily deduced that the function has to be strictly increasing. Let us prove that formally. Assume the contrary, that for some two real numbers $x_1<x_2$ we have $f(x_1)\geq f(x_2)$. The continuity on $[0,x_1]$ implies that there is some c such that $f(c)=f(x_2)$, which contradicts the injectivity of f. Now if $x<f(x)$, we get $f(x)<f(f(x))$ etc. $x<f(n)(x)=x$. Similarly we get a contradiction if we assume that $x>f(x)$. Hence for each $x\in[0,1]$ we must have $f(x)=x$. △

Problem 18. *Find all functions $f:(0,+\infty)\to(0,+\infty)$ that satisfy $f(f(x)+y)=xf(1+xy)$ for all $x,y\in(0,+\infty)$.*

Solution. Clearly $f(x)=\frac{1}{x}$ is one solution to the functional equation. Let us prove that the function is non-increasing. Assume the contrary that for some $0<x<y$ we have $0<f(x)<f(y)$. We will consider the expression of the form $z=\frac{yf(y)-xf(x)}{y-x}$ since it is positive and bigger then $f(y)$. We first plug $(x,z-f(y))$ instead of (x,y) in the original equation, then we plug $z-f(x)$ instead of y, we get $x=y$, which is a contradiction. Hence the function is non-decreasing.

Let us prove that $f(1)=1$. Let $f(1)\neq 1$. Substituting $x=1$ we get $f(f(1)+y)=f(1+y)$, hence $f(u+|f(1)-1|)=f(u)$ for $u>1$. Therefore the function is periodic on the interval $(1,+\infty)$, and since it is monotone it is constant. However we then conclude that the left-hand side of the original equation constant and the right-hand side is not. Thus we must have $f(1)=1$. Let us prove that $f(x)=\frac{1}{x}$ for $x>1$. Indeed for $y=1-\frac{1}{x}$ the given equality gives $f\Big(f(x)-\frac{1}{x}\Big)=xf(x)$. If $f(x)>\frac{1}{x}$ we have $f\Big(f(x)-\frac{1}{x}+1\Big)\leq f(1)=1$ and $xf(x)>1$. If $f(x)<\frac{1}{x}$ we have $f\Big(f(x)-\frac{1}{x}+1\Big)\geq f(1)=1$, and $xf(x)<1$. Hence $f(x)=\frac{1}{x}$. If $x<1$, plugging $y=\frac{1}{x}$ we get

$$f\Big(f(x)+\frac{1}{x}\Big)=xf(2)=\frac{x}{2},$$

and since $\frac{1}{x}\geq 1$, we get $f(x)+\frac{1}{x}=\frac{2}{x}$, i.e. $f(x)=\frac{1}{x}$ in this case, too. This means that $f(x)=\frac{1}{x}$ for all positive real numbers x. △

Problem 19. *(Bulgaria 1998) Prove that there is no function $f:\mathbb{R}^+\to\mathbb{R}^+$ such that $f(x)^2\geq f(x+y)(f(x)+y)$ for every two positive real numbers x and y.*

Solution. The common idea for the problems of this type is to prove that $f(y)<0$ for some $y>0$ which will lead us to the obvious contradiction. We can also see that it is sufficient to prove that $f(x)-f(x+1)\geq c>0$, for every x because the simple addition gives $f(x)-f(x+m)\geq mc$. For sufficiently large m this implies $f(x+m)<0$. Hence our goal is finding c such that $f(x)-f(x+1)\geq c$, for every x. Assume that such function exists. From the given inequality we get $f(x)-f(x+y)\geq \frac{f(x+y)y}{f(x)}$ and the function is obviously decreasing. Also from the given equality we can conclude that

$$f(x)-f(x+y)\geq\frac{f(x)y}{f(x)+y}.$$

Let n be a natural number such that $f(x+1)n\geq 1$ (such number clearly exists). Notice that for $0\leq k\leq n-1$ the following inequality holds

$$f\Big(x+\frac{k}{n}\Big)-f\Big(x+\frac{k+1}{n}\Big)\geq\frac{f\Big(x+\frac{k}{n}\Big)\frac{1}{n}}{f\Big(x+\frac{k}{n}\Big)+\frac{1}{n}}\geq\frac{1}{2n},$$

and adding similar realitions for all described k yields $f(x)-f(x+1)\geq\frac{1}{2}$ which is a contradiction. $\triangle$

Problem 20. *Let $f:\mathbb{N}\to\mathbb{N}$ be a function satisfying*

$$f(1)=2,\quad f(2)=1,\quad f(3n)=3f(n),\quad f(3n+1)=3f(n)+2,\quad f(3n+2)=3f(n)+1.$$

Find the number of integers $n\leq 2006$ for which $f(n)=2n$.

Solution. This is a typical problem in which the numbers should be considered in some base different than 10. For this situation the base 3 is doing the job. Let us calculate $f(n)$ for $n\leq 8$ in an attempt to guess the solution. Clearly the given equation can have only one solution.

$$f((1)_3)=(2)_3,\ f((2)_3)=(1)_3,\ f((10)_3)=6=(20)_3,\ f((11)_3)=8=(22)_3,$$

$$f((12)_3)=7=(21)_3,\ f((20)_3)=3=(10)_3,\ f((21)_3)=5=(12)_3,\ f((22)_3)=4=(11)_3.$$

Now we see that $f(n)$ is obtained from n by changing each digit 2 by 1, and conversely. This can be now easily shown by induction. It is clear that $f(n)=2n$ if and only if in the system with base 3 n doesn't contain any digit 1 (because this would imply $f(n)<2n$). Now it is easy to count the number of such n's. The answer is 127. $\triangle$

Problem 21. *(BMO 2003, shortlist) Find all possible values for $f\Big(\frac{2004}{2003}\Big)$ if $f:\mathbb{Q}\to[0,+\infty)$ is the function satisfying the conditions:*

(i) $f(xy)=f(x)f(y)$ for all $x,y\in\mathbb{Q}$;

(ii) $f(x)\leq 1\Rightarrow f(x+1)\leq 1$ for all $x\in\mathbb{Q}$;

(iii) $f\Big(\frac{2003}{2002}\Big)=2$.

Solution. Notice that from (i) and (ii) we conclude that $f(x)>0$, for every rational x. Now (i) implies that for $x=y=1$ we get $f(1)=0$ and similarly for $x=y=-1$ we get $f(-1)=1$. By induction $f(x)\leq 1$ for every integer x. For $f(x)\leq f(y)$ from $f\Big(\frac{y}{x}\Big)f(y)=f(x)$ we have that $f\Big(\frac{y}{x}\Big)\leq 1$, and according to (ii) $f\Big(\frac{y}{x}+1\Big)\leq 1$. This implies

$$f(x+y)=f\Big(\frac{y}{x}+1\Big)f(x)\leq f(x),$$

hence $f(x+y)\leq\max\{f(x),f(y)\}$, for every $x,y\in\mathbb{Q}$. Now you might wonder how did we get this idea. There is one often neglected fact that for every two relatively prime numbers u and v, there are integers a and b such that $au+bv=1$. What is all of this good for? We got that $f(1)=1$, and we know that $f(x)\leq 1$ for all $x\in\mathbb{Z}$ and since 1 is the maximum of the function on $\mathbb{Z}$ and since we have the previous inequality our goal is to show that the value of the function is 1 for a bigger class of integers. We will do this for prime numbers. If for every prime p we have $f(p)=1$ then $f(x)=1$ for every integer implying $f(x)\equiv 1$ which contradicts (iii). Assume therefore that $f(p)\neq 1$ for some $p\in\mathbb{P}$. There are a and b such that $ap+bq=1$ implying $f(1)=f(ap+bq)\leq\max\{f(ap),f(bq)\}$. Now we must have $f(bq)=1$ implying that $f(q)=1$ for every other prime number q. From (iii) we have

$$f\Big(\frac{2003}{2002}\Big)=\frac{f(2003)}{f(2)f(7)f(11)f(13)}=2,$$

hence only one of the numbers $f(2),f(7),f(11),f(13)$ is equal to $1/2$. Thus $f(3)=f(167)=f(2003)$ giving:

$$f\Big(\frac{2004}{2003}\Big)=\frac{f(2)^2f(3)f(167)}{f(2003)}=f(2)^2.$$

If $f(2) = 1/2$ then $f\Big(\frac{2003}{2002}\Big) = \frac{1}{4}$, otherwise it is 1.

It remains to construct one function for each of the given values. For the first value it is the multiplicative function taking the value $1/2$ at the point 2, and 1 for all other prime numbers; in the second case it is a the multiplicative function that takes the value $1/2$ at, for example, 7 and takes 1 at all other prime numbers. For these functions we only need to verify the condition (ii), but that is also very easy to verify. △

Problem 22. *Let $I = [0,1]$, $G = I \times I$ and $k \in \mathbb{N}$. Find all $f : G \to I$ such that for all $x,y,z \in I$ the following statements hold:*

(i) $f(f(x,y),z) = f(x, f(y,z))$;

(ii) $f(x,1) = x$, $f(x,y) = f(y,x)$;

(ii) $f(zx,zy) = z^k f(x,y)$ *for every $x,y,z \in I$, where k is a fixed real number.*

Solution. The function of several variables appears in this problem. In most cases we use the same methods as in the case of a single-variable functions. From the condition (ii) we get $f(1,0) = f(0,1) = 0$, and from (iii) we get $f(0,x) = f(x,0) = x^k f(1,0) = 0$. This means that f is entirely defined on the edge of the region G. Assume therefore that $0 < x \leq y < 1$. Notice that the condition (ii) gives the value for one class of pairs from G and that each pair in G can be reduced to one of the members of the class. This implies

$$f(x,y) = f(y,x) = y^k f\Big(1, \frac{x}{y}\Big) = y^{k-1}x.$$

This can be written as $f(x,y) = \min(x,y)(\max(u,v))^{k-1}$ for all $0 < x,y < 1$. Let us find all possible values for k. Let $0 < x \leq \frac{1}{2} \leq y < 1$. From the condition (i), and the already obtained results we get

$$f\Big(f\Big(x,\frac{1}{2}\Big),y\Big) = f\Big(x\Big(\frac{1}{2}\Big)^{k-1},y\Big) = f\Big(x, f\Big(\frac{1}{2}\Big)\Big) = f\Big(x, \frac{1}{2}y^{k-1}\Big).$$

Let us now consider $x \leq 2^{k-1}y$ in order to simplify the expression to the form $f\Big(x, \frac{1}{2}y^{k-1}\Big) = x\Big(\frac{y}{2}\Big)^{k-1}$, and if we take x for which $2x \leq y^{k-1}$ we get $k-1 = (k-1)^2$, i.e. $k = 1$ or $k = 2$. For $k = 1$ the solution is $f(x,y) = \min(x,y)$, and for $k = 2$ the solution is $f(x,y) = xy$. It is easy to verify that both solutions satisfy the given conditions. △

Problem 23. *(APMO 1989) Find all strictly increasing functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) + g(x) = 2x,$$

where g is the inverse of f.

Solution. Clearly every function of the form $x + d$ is the solution of the given equation. Another useful idea appears in this problem. Namely denote by S_d the the set of all numbers x for which $f(x) = x + d$. Our goal is to prove that $S_d = \mathbb{R}$. Assume that S_d is non-empty. Let us prove that for $x \in S_d$ we have $x + d \in S_d$ as well. Since $f(x) = x + d$, according to the definition of the inverse function we have $g(x+d) = x$, and the given equation implies $f(x+d) = x + 2d$, i.e. $x + d \in S_d$. Let us prove that the sets $S_{d'}$ are empty, where $d' < d$. From the above we have that each of those sets is infinite, i.e. if x belongs to some of them, then each $x + kd$ belongs to it as well. Let us use this to get the contradiction. More precisely we want to prove that if $x \in S_d$ and $x \leq y \leq x + (d - d')$, then $y \notin S_{d'}$. Assume the contrary. From the monotonicity we have $y + d' = f(y) \geq f(x) = x + d$, which is a contradiction to our assumption. By further induction we prove that every y satisfying

$$x + k(d - d') \leq y < x + (k+1)(d - d'),$$

can't be a member of $S_{d'}$. However this is a contradiction with the previously established properties of the sets S_d and $S_{d'}$. Similarly if $d' > d$ switching the roles of d and d' gives a contradiction.

Simple verification shows that each $f(x) = x + d$ satisfies the given functional equation. △

Problem 24. *Find all functions* $h : \mathbb{N} \to \mathbb{N}$ *that satisfy*

$$h(h(n)) + h(n+1) = n+2.$$

Solution. Notice that we have both $h(h(n))$ and $h(n+1)$, hence it is not possible to form a recurrent equation. We have to use another approach to this problem. Let us first calculate $h(1)$ and $h(2)$. Setting $n = 1$ gives $h(h(1)) + h(2) = 3$, therefore $h(h(1)) \leq 2$ and $h(2) \leq 2$. Let us consider the two cases:

1° $h(2) = 1$. Then $h(h(1)) = 2$. Plugging $n = 2$ in the given equality gives $4 = h(h(2)) + h(3) = h(1) + h(3)$. Let $h(1) = k$. It is clear that $k \neq 1$ and $k \neq 2$, and that $k \leq 3$. This means that $k = 3$, hence $h(3) = 1$. However from $2 = h(h(1)) = h(3) = 1$ we get a contradiction. This means that there are no solutions in this case.

2° $h(2) = 2$. Then $h(h(1)) = 1$. From the equation for $n = 2$ we get $h(3) = 2$. Setting $n = 3, 4, 5$ we get $h(4) = 3, h(5) = 4, h(6) = 4$, and by induction we easily prove that $h(n) \geq 2$, for $n \geq 2$. This means that $h(1) = 1$. Clearly there is at most one function satisfying the given equality. Hence it is enough to guess some function and prove that it indeed solves the equation (induction or something similar sounds fine). The solution is

$$h(n) = \lfloor n\alpha \rfloor + 1,$$

where $\alpha = \dfrac{-1+\sqrt{5}}{2}$ (this constant can be easily found $\alpha^2 + \alpha = 1$). Proof that this is a solution uses some properties of the integer part (although it is not completely trivial). △

Problem 25. *(IMO 2004, shortlist) Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying the equality*

$$f(x^2 + y^2 + 2f(xy)) = f(x+y)^2.$$

Solution. Let us make the substitution $z = x + y$, $t = xy$. Given $z, t \in \mathbb{R}$, x, y are real if and only if $4t \leq z^2$. Define $g(x) = 2(f(x) - x)$. Now the given functional equation transforms into

$$f\left(z^2 + g(t)\right) = (f(z))^2 \quad \text{for all } t, z \in \mathbb{R} \text{ with } z^2 \geq 4t. \tag{8}$$

Let us set $c = g(0) = 2f(0)$. Substituting $t = 0$ into (8) gives us

$$f(z^2 + c) = (f(z))^2 \quad \text{for all } z \in \mathbb{R}. \tag{9}$$

If $c < 0$, then taking z such that $z^2 + c = 0$, we obtain from (9) that $f(z)^2 = c/2$, which is impossible; hence $c \geq 0$. We also observe that

$$x > c \quad \text{implies} \quad f(x) \geq 0. \tag{10}$$

If g is a constant function, we easily find that $c = 0$ and therefore $f(x) = x$, which is indeed a solution.

Suppose g is nonconstant, and let $a, b \in \mathbb{R}$ be such that $g(a) - g(b) = d > 0$. For some sufficiently large K and each $u, v \geq K$ with $v^2 - u^2 = d$ the equality $u^2 + g(a) = v^2 + g(b)$ by (8) and (10) implies $f(u) = f(v)$. This further leads to $g(u) - g(v) = 2(v - u) = \frac{d}{u+\sqrt{u^2+d}}$. Therefore every value from some suitably chosen segment $[\delta, 2\delta]$ can be expressed as $g(u) - g(v)$, with u and v bounded from above by some M.

Consider any x,y with $y > x \geq 2\sqrt{M}$ and $\delta < y^2 - x^2 < 2\delta$. By the above considerations, there exist $u,v \leq M$ such that $g(u) - g(v) = y^2 - x^2$, i.e., $x^2 + g(u) = y^2 + g(v)$. Since $x^2 \geq 4u$ and $y^2 \geq 4v$, (8) leads to $f(x)^2 = f(y)^2$. Moreover, if we assume w.l.o.g. that $4M \geq c^2$, we conclude from (10) that $f(x) = f(y)$. Since this holds for any $x,y \geq 2\sqrt{M}$ with $y^2 - x^2 \in [\delta, 2\delta]$, it follows that $f(x)$ is eventually constant, say $f(x) = k$ for $x \geq N = 2\sqrt{M}$. Setting $x > N$ in (9) we obtain $k^2 = k$, so $k = 0$ or $k = 1$.

By (9) we have $f(-z) = \pm f(z)$, and thus $|f(z)| \leq 1$ for all $z \leq -N$. Hence $g(u) = 2f(u) - 2u \geq -2 - 2u$ for $u \leq -N$, which implies that g is unbounded. Hence for each z there exists t such that $z^2 + g(t) > N$, and consequently $f(z)^2 = f(z^2 + g(t)) = k = k^2$. Therefore $f(z) = \pm k$ for each z.

If $k = 0$, then $f(x) \equiv 0$, which is clearly a solution. Assume $k = 1$. Then $c = 2f(0) = 2$ (because $c \geq 0$), which together with (10) implies $f(x) = 1$ for all $x \geq 2$. Suppose that $f(t) = -1$ for some $t < 2$. Then $t - g(t) = 3t + 2 > 4t$. If also $t - g(t) \geq 0$, then for some $z \in \mathbb{R}$ we have $z^2 = t - g(t) > 4t$, which by (8) leads to $f(z)^2 = f(z^2 + g(t)) = f(t) = -1$, which is impossible. Hence $t - g(t) < 0$, giving us $t < -2/3$. On the other hand, if X is any subset of $(-\infty, -2/3)$, the function f defined by $f(x) = -1$ for $x \in X$ and $f(x) = 1$ satisfies the requirements of the problem.

To sum up, the solutions are $f(x) = x$, $f(x) = 0$ and all functions of the form

$$f(x) = \begin{cases} 1, & x \notin X, \\ -1, & x \in X, \end{cases}$$

where $X \subset (-\infty, -2/3)$. $\triangle$

4 Problems for Independent Study

Most of the ideas for solving the problems below are already mentioned in the introduction or in the section with solved problems. The difficulty of the problems vary as well as the range of ideas used to solve them. Before solving the problems we highly encourage you to first solve (or look at the solutions) the problems from the previous section. Some of the problems are quite difficult.

1. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ that satisfy $f(x+y) = f(x) + f(y) + xy$.

2. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ for which we have $f(0) = 1$ and $f(f(n)) = f(f(n+2)+2) = n$, for every natural number n.

3. Find all functions $f : \mathbb{N} \to \mathbb{N}$ for which $f(n)$ is a square of an integer for all $n \in \mathbb{N}$, and that satisfy $f(m+n) = f(m) + f(n) + 2mn$ for all $m,n \in \mathbb{N}$.

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f((x-y)^2) = f(x)^2 - 2xf(y) + y^2$.

5. Let $n \in \mathbb{N}$. Find all monotone functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) = f(x) + y^n.$$

6. (USA 2002) Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equality $f(x^2 - y^2) = xf(x) - yf(y)$.

7. (Mathematical High Schol, Belgrade 2004) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(m) + f(n)) = m + n$ for every two natural numbers m and n.

8. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(xy) = xf(y) + yf(x)$.

9. (IMO 1983, problem 1) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

   (i) $f(xf(y)) = yf(x)$, for all $x,y \in \mathbb{R}$;

   (ii) $f(x) \to 0$ as $x \to +\infty$.

10. Let $f : \mathbb{N} \to \mathbb{N}$ be strictly increasing function that satisfies $f(f(n)) = 3n$ for every natural number n. Determine $f(2006)$.

11. (IMO 1989, shortlist) Let $0 < a < 1$ be a real number and f continuous function on $[0,1]$ which satisfies $f(0) = 0$, $f(1) = 1$, and
$$f\Big(\frac{x+y}{2}\Big) = (1-a)f(x) + af(y),$$
for every two real numbers $x, y \in [0,1]$ such that $x \le y$. Determine $f\Big(\frac{1}{7}\Big)$.

12. (IMO 1996, shortlist) Let $f : \mathbb{R} \to \mathbb{R}$ be the function such that $|f(x)| \le 1$ and
$$f\Big(x+\frac{13}{42}\Big) + f(x) = f\Big(x+\frac{1}{6}\Big) + f\Big(x+\frac{1}{7}\Big).$$
Prove that f is periodic.

13. (BMO 2003, problem 3) Find all functions $f : \mathbb{Q} \to \mathbb{R}$ that satisfy:
    (i) $f(x+y) - yf(x) - xf(y) = f(x)f(y) - x - y + xy$ for every $x, y \in \mathbb{Q}$;
    (ii) $f(x) = 2f(x+1) + 2 + x$, for every $x \in \mathbb{Q}$;
    (iii) $f(1) + 1 > 0$.

14. (IMO 1990, problem 4) Determine the function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that
$$f(xf(y)) = \frac{f(x)}{y}, \quad \text{for all } x, y \in \mathbb{Q}^+.$$

15. (IMO 2002, shortlist) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(f(x)+y) = 2x + f(f(y) - x).$$

16. (Iran 1997) Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function such that for all positive real numbers x and y:
$$f(x+y) + f(f(x)+f(y)) = f(f(x+f(y)) + f(y+f(x))).$$
Prove that $f(f(x)) = x$.

17. (IMO 1992, problem 2) Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(x^2 + f(y)) = y + f(x)^2$ for all $x, y \in \mathbb{R}$.

18. (IMO 1994, problem 5) Let S be the set of all real numbers strictly greater than -1. Find all functions $f : S \to S$ that satisfy the following two conditions:
    (i) $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all $x, y \in S$;
    (ii) $\dfrac{f(x)}{x}$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$.

19. (IMO 1994, shortlist) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that
$$f(x)f(y) = y^\alpha f(x/2) + x^\beta f(y/2), \quad \text{for all } x, y \in \mathbb{R}^+.$$

20. (IMO 2002, problem 5) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$(f(x)+f(z))(f(y)+f(t)) = f(xy - zt) + f(xt + yz).$$

21. (Vietnam 2005) Find all values for a real parameter α for which there exists exactly one function $f:\mathbb{R}\to\mathbb{R}$ satisfying
$$f(x^2+y+f(y))=f(x)^2+\alpha\cdot y.$$

22. (IMO 1998, problem 3) Find the least possible value for $f(1998)$ where $f:\mathbb{N}\to\mathbb{N}$ is a function that satisfies
$$f(n^2f(m))=mf(n)^2.$$

23. Does there exist a function $f:\mathbb{N}\to\mathbb{N}$ such that
$$f(f(n-1))=f(n+1)-f(n)$$
for each natural number n?

24. (IMO 1987, problem 4) Does there exist a function $f:\mathbb{N}_0\to\mathbb{N}_0$ such that $f(f(n))=n+1987$?

25. Assume that the function $f:\mathbb{N}\to\mathbb{N}$ satisfies $f(n+1)>f(f(n))$, for every $n\in\mathbb{N}$. Prove that $f(n)=n$ for every n.

26. Find all functions $f:\mathbb{N}_0\to\mathbb{N}_0$, that satisfy:
    (i) $2f(m^2+n^2)=f(m)^2+f(n)^2$, for every two natural numbers m and n;
    (ii) If $m\geq n$ then $f(m^2)\geq f(n^2)$.

27. Find all functions $f:\mathbb{N}_0\to\mathbb{N}_0$ that satisfy:
    (i) $f(2)=2$;
    (ii) $f(mn)=f(m)f(n)$ for every two relatively prime natural numbers m and n;
    (iii) $f(m)<f(n)$ whenever $m<n$.

28. Find all functions $f:\mathbb{N}\to[1,\infty)$ that satisfy conditions (i) and (ii9) of the previous problem and the condition (ii) is modified to require the equality for every two natural numbers m and n.

29. Given a natural number k, find all functions $f:\mathbb{N}_0\to\mathbb{N}_0$ for which
$$f(f(n))+f(n)=2n+3k,$$
for every $n\in\mathbb{N}_0$.

30. (Vijetnam 2005) Find all functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy $f(f(x-y))=f(x)f(y)-f(x)+f(y)-xy$.

31. (China 1996) The function $f:\mathbb{R}\to\mathbb{R}$ satisfy $f(x^3+y^3)=(x+y)\Big(f(x)^2-f(x)f(y)+f(y)^2\Big)$, for all real numbers x and y. Prove that $f(1996x)=1996f(x)$ for every $x\in\mathbb{R}$.

32. Find all functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy:
    (i) $f(x+y)=f(x)+f(y)$ for every two real numbers x and y;
    (ii) $f\left(\frac{1}{x}\right)=\frac{f(x)}{x^2}$ for $x\neq 0$.

33. (IMO 1989, shortlist) A function $f:\mathbb{Q}\to\mathbb{R}$ satisfy the following conditions:
    (i) $f(0)=0$, $f(\alpha)>0$ za $\alpha\neq 0$;

(ii) $f(\alpha\beta) = f(\alpha)f(\beta)$ i $f(\alpha+\beta) \le f(\alpha) + f(\beta)$, for all $\alpha, \beta \in \mathbb{Q}$;

(iii) $f(m) \le 1989$ za $m \in \mathbb{Z}$.

Prove that $f(\alpha+\beta) = \max\{f(\alpha), f(\beta)\}$ whenever $f(\alpha) \neq f(\beta)$.

34. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for every two real numbers $x \neq y$ the equality

$$f\left(\frac{x+y}{x-y}\right) = \frac{f(x)+f(y)}{f(x)-f(y)}$$

is satisfied.

35. Find all functions $f:\mathbb{Q}^+\to\mathbb{Q}^+$ satisfying:

(i) $f(x+1) = f(x)+1$ for all $x \in \mathbb{Q}^+$;

(ii) $f(x^3) = f(x)^3$ for all $x \in \mathbb{Q}^+$.

36. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy the equality

$$f(x+y) + f(xy) = f(x) + f(y) + f(xy+1).$$

37. Find all continuous functions $f, g, h, k:\mathbb{R}\to\mathbb{R}$ that satisfy the equality

$$f(x+y) + g(x-y) = 2h(x) + 2k(y).$$

38. (IMO 1996, shortlist) Find all functions $f:\mathbb{N}_0\to\mathbb{N}_0$ such that

$$f(m+f(n)) = f(f(m)) + f(n).$$

39. (IMO 1995, shortlist) Does there exist a function $f:\mathbb{R}\to\mathbb{R}$ satisfying the conditions:

(i) There exists a positive real number M such that $-M \le f(x) \le M$ for all $x \in \mathbb{R}$;

(ii) $f(1) = 1$;

(iii) If $x \neq 0$ then $f\left(x + \frac{1}{x^2}\right) = f(x) + \left[f\left(\frac{1}{x}\right)\right]^2$?

40. (Belarus) Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy

$$f(f(x)) = f(x) + 2x.$$

41. Prove that if the function $f:\mathbb{R}^+\to\mathbb{R}$ satisfy the equality

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{2xy}{x+y}\right) = f(x) + f(y),$$

the it satisfy the equality $2f(\sqrt{xy}) = f(x) + f(y)$ as well.

42. Find all continuous functions $f:(0,\infty)\to(0,\infty)$ that satisfy

$$f(x)f(y) = f(xy) + f(x/y).$$

43. Prove that there is no function $f:\mathbb{R}\to\mathbb{R}$ that satisfy the inequality $f(y) > (y-x)f(x)^2$, for every two real numbers x and y.

44. (IMC 2001) Prove that there doesn't exist a function $f:\mathbb{R}\to\mathbb{R}$ for which $f(0)>0$ and
$$f(x+y)\geq f(x)+yf(f(x)).$$

45. (Romania 1998) Find all functions $u:\mathbb{R}\to\mathbb{R}$ for which there exists a strictly monotone function $f:\mathbb{R}\to\mathbb{R}$ such that
$$f(x+y)=f(x)u(y)+f(y),\quad \forall x,y\in\mathbb{R}.$$

46. (Iran 1999) Find all functions $f:\mathbb{R}\to\mathbb{R}$ for which
$$f(f(x)+y)=f(x^2-y)+4f(x)y.$$

47. (IMO 1988, problem 3) A function $f:\mathbb{N}\to\mathbb{N}$ satisfies:
    (i) $f(1)=1$, $f(3)=3$;
    (ii) $f(2n)=f(n)$;
    (iii) $f(4n+1)=2f(2n+1)-f(n)$ and $f(4n+3)=3f(2n+1)-2f(n)$,

    for every natural number $n\in\mathbb{N}$. Find all natural numbers $n\leq 1998$ such that $f(n)=n$.

48. (IMO 2000, shortlist) Given a function $F:\mathbb{N}_0\to\mathbb{N}_0$, assume that for $n\geq 0$ the following relations hold:
    (i) $F(4n)=F(2n)+F(n)$;
    (ii) $F(4n+2)=F(4n)+1$;
    (iii) $F(2n+1)=F(2n)+1$.

    Prove that for every natural number m, the number of positive integers n such that $0\leq n<2^m$ and $F(4n)=F(3n)$ is equal to $F(2^{m+1})$.

49. Let $f:\mathbb{Q}\times\mathbb{Q}\to\mathbb{Q}^+$ be a function satisfying
$$f(xy,z)=f(x,z)f(y,z),\quad f(z,xy)=f(z,x)f(z,y),\quad f(x,1-x)=1,$$
for all rational numbers x,y,z. Prove that $f(x,x)=1$, $f(x,-x)=1$, and $f(x,y)f(y,x)=1$.

50. Find all functions $f:\mathbb{N}\times\mathbb{N}\to\mathbb{R}$ that satisfy
$$f(x,x)=x,\quad f(x,y)=f(y,x),\quad (x+y)f(x,y)=yf(x,x+y).$$

Polynomials in One Variable

Dušan Djukić

Contents

1 General Properties

A *Monomial* in variable x is an expression of the form cx^k, where c is a constant and k a nonnegative integer. Constant c can be e.g. an integer, rational, real or complex number.

A *Polynomial* in x is a sum of finitely many monomials in x. In other words, it is an expression of the form

$$P(x) = a_n x^n + a_{n-1} + \cdots + a_1 x + a_0. \qquad (*)$$

If only two or three of the above summands are nonzero, P is said to be a *binomial* and *trinomial*, respectively.

The constants $a_0, \dots, a_n$ in $(*)$ are the *coefficients* of polynomial P. The set of polynomials with the coefficients in set A is denoted by $A[x]$ - for instance, $\mathbb{R}[x]$ is the set of polynomials with real coefficients.

We can assume in $(*)$ w.l.o.g. that $a_n \neq 0$ (if $a_n = 0$, the summand $a_n x^n$ can be erased without changing the polynomial). Then the exponent n is called the *degree* of polynomial P and denoted by $\deg P$. In particular, polynomials of degree one, two and three are called *linear*, *quadratic* and *cubic*. A nonzero constant polynomial has degree 0, while the zero-polynomial $P(x) \equiv 0$ is assigned the degree $-\infty$ for reasons soon to become clear.

Example 1. *$P(x) = x^3(x+1) + (1-x^2)^2 = 2x^4 + x^3 - 2x^2 + 1$ is a polynomial with integer coefficients of degree 4.*

$Q(x) = 0x^2 - \sqrt{2}x + 3$ is a linear polynomial with real coefficients.

$R(x) = \sqrt{x^2} = |x|$, $S(x) = \frac{1}{x}$ and $T(x) = \sqrt{2x+1}$ are not polynomials.

Polynomials can be added, subtracted or multiplied, and the result will be a polynomial too:

$$A(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad B(x) = b_0 + b_1 x + \cdots + b_m x^m$$
$$A(x) \pm B(x) = (a_0 - b_0) + (a_1 - b_1)x + \cdots,$$
$$A(x)B(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + \cdots + a_n b_m x^{m+n}.$$

The behavior of the degrees of the polynomials under these operations is clear:

Theorem 1. *If A and B are two polynomials then:*

(i) $\deg(A \pm B) \le \max(\deg A, \deg B)$*, with the equality if* $\deg A \neq \deg B$.

(ii) $\deg(A \cdot B) = \deg A + \deg B$. □

The conventional equality $\deg 0 = -\infty$ actually arose from these properties of degrees, as else the equality (ii) would not be always true.

Unlike a sum, difference and product, a quotient of two polynomials is not necessarily a polynomial. Instead, like integers, they can be divided with a residue.

Theorem 2. *Given polynomials A and $B \neq 0$, there are unique polynomials Q (quotient) and R (residue) such that*

$$A = BQ + R \quad \text{and} \quad \deg R < \deg B.$$

Proof. Let $A(x) = a_n x^n + \cdots + a_0$ and $B(x) = b_k x^k + \cdots + b_0$, where $a_n b_k \neq 0$. Assume k is fixed and use induction on n. For $n < k$ the statement is trivial. Suppose that $n = N \geq k$ and that the statement is true for $n < N$. Then $A_1(x) = A(x) - \frac{a_n}{b_k} x^{n-k} B(x)$ is a polynomial of degree less than n (for its coefficient at x^n iz zero); hence by the inductive assumption there are unique polynomials Q_1 and R such that $A_1 = BQ_1 + R$ and $\deg R$. But this also implies

$$A = BQ + R, \quad \text{where} \quad Q(x) = \frac{a_n}{b_k} x^{n-k} + Q_1(x)\,. \ \square$$

Example 2. *The quotient upon division of $A(x) = x^3 + x^2 - 1$ by $B(x) = x^2 - x - 3$ is $x + 2$ with the residue $5x + 5$, as*

$$\frac{x^3 + x^2 - 1}{x^2 - x - 3} = x + 2 + \frac{5x + 5}{x^2 - x - 3}\,.$$

We say that polynomial A is *divisible* by polynomial B if the remainder R when A is divided by B equal to 0, i.e. if there is a polynomial Q such that $A = BQ$.

Theorem 3 (Bezout's theorem). *Polynomial $P(x)$ is divisible by binomial $x - a$ if and only if $P(a) = 0$.*

Proof. There exist a polynomial Q and a constant c such that $P(x) = (x - a)Q(x) + c$. Here $P(a) = c$, making the statement obvious. □

Number a is a *zero (root)* of a given polynomial $P(x)$ if $P(a) = 0$, i.e. $(x - a) \mid P(x)$.

To determine a zero of a polynomial f means to solve the equation $f(x) = 0$. This is not always possible. For example, it is known that finding the exact values of zeros is impossible in general when f is of degree at least 5. Nevertheless, the zeros can always be computed with an arbitrary precision. Specifically, $f(a) < 0 < f(b)$ implies that f has a zero between a and b.

Example 3. *Polynomial $x^2 - 2x - 1$ has two real roots: $x_{1,2} = 1 \pm \sqrt{2}$.*

Polynomial $x^2 - 2x + 2$ has no real roots, but it has two complex roots: $x_{1,2} = 1 \pm i$.

Polynomial $x^5 - 5x + 1$ has a zero in the interval $[1.44, 1.441]$ which cannot be exactly computed.

More generally, the following simple statement holds.

Theorem 4. *If a polynomial P is divisible by a polynomial Q, then every zero of Q is also a zero of P.* □

The converse does not hold. Although every zero of x^2 is a zero of x, x^2 does not divide x.

Problem 1. *For which n is the polynomial $x^n + x - 1$ divisible by a) $x^2 - x + 1$, b) $x^3 - x + 1$?*

Solution. a) The zeros of polynomial x^2-x+1 are $\varepsilon_{1,2}=\frac{1\pm i\sqrt{3}}{2}$. If x^2-x+1 divides x^n+x-1, then $\varepsilon_{1,2}$ are zeros of polynomial x^n+x-1, so $\varepsilon_i^n=1-\varepsilon_i=\varepsilon_i^{-1}$. Since $\varepsilon^k=1$ if and only if $6\mid k$, the answer is $n=6i-1$.

b) If $f(x)=x^3-x+1$ divides x^n+x-1, then it also divides x^n+x^3. This means that every zero of $f(x)$ satisfies $x^{n-3}=-1$; in particular, each zero of f has modulus 1. However, $f(x)$ has a zero between -2 and -1 (for $f(-2)<0<f(-1)$) which is obviously not of modulus 1. Hence there is no such n. △

Every nonconstant polynomial with complex coefficients has a complex root. We shall prove this statement later; until then we just believe.

The following statement is analogous to the unique factorization theorem in arithmetics.

Theorem 5. *Polynomial $P(x)$ of degree $n>0$ has a unique representation of the form*

$$P(x)=c(x-x_1)(x-x_2)\cdots(x-x_n),$$

not counting the ordering, where $c\neq 0$ and $x_1,\dots,x_n$ are complex numbers, not necessarily distinct.

Therefore, $P(x)$ has at most $\deg P=n$ different zeros.

Proof. First we show the uniqueness. Suppose that

$$P(x)=c(x-x_1)(x-x_2)\cdots(x-x_n)=d(x-y_1)(x-y_2)\cdots(x-y_n).$$

Comparing the leading coefficients yields $c=d$. We may assume w.l.o.g. that there are no i,j for which $x_i=y_j$ (otherwise the factor $x-x_i$ can be canceled on both sides). Then $P(x_1)=0$. On the other hand, $P(x_1)=d(x_1-y_1)\cdots(x_1-y_n)\neq 0$, a contradiction.

The existence is shown by induction on n. The case $n=1$ is clear. Let $n>1$. The polynomial $P(x)$ has a complex root, say x_1. By Bezout's theorem, $P(x)=(x-x_1)P_1(x)$ for some polynomial P_1 of degree $n-1$. By the inductive assumption there exist complex numbers $x_2,\dots,x_n$ for which $P_1(x)=c(x-x_2)\cdots(x-x_n)$, which also implies $P(x)=c(x-x_1)\cdots(x-x_n)$. □

Corollary. If polynomials P and Q has degrees not exceeding n and coincide at $n+1$ different points, then they are equal.

Grouping equal factors yields the *canonical representation*:

$$P(x)=c(x-a_1)^{\alpha_1}(x-a_2)^{\alpha_2}\cdots(x-a_k)^{\alpha_k},$$

where α_i are natural numbers with $\alpha_1+\cdots+\alpha_k=n$. The exponent α_i is called the *multiplicity* of the root a_i. It is worth emphasizing that:

Theorem 6. *Polynomial of n-th degree has exactly n complex roots counted with their multiplicities.* □

We say that two polynomials Q and R are *coprime* if they have no roots in common; Equivalently, there is no nonconstant polynomial dividing them both, in analogy with coprimeness of integers. The following statement is a direct consequence of the previous theorem:

Theorem 7. *If a polynomial P is divisible by two coprime polynomials Q and R, then it is divisible by $Q\cdot R$.* □

Remark: This can be shown without using the existence of roots. By the Euclidean algorithm applied on polynomials there exist polynomials K and L such that $KQ+LR=1$. Now if $P=QS=RT$ for some polynomials R,S, then $R(KT-LS)=KQS-LRS=S$, and therefore $R\mid S$ and $QR\mid QS=P$.

If polynomial $P(x)=x^n+\cdots+a_1x+a_0$ with real coefficients has a complex zero ξ, then $P(\overline{\xi})=\overline{\xi^n}+\cdots+a_1\overline{\xi}+a_0=\overline{P(\xi)}=0$. Thus:

Theorem 8. *If ξ is a zero of a real polynomial $P(x)$, then so is $\overline{\xi}$.* □

In the factorization of a real polynomial $P(x)$ into linear factors we can group conjugated complex zeros:

$$P(x) = (x-r_1)\cdots(x-r_k)(x-\xi_1)(x-\overline{\xi_1})\cdots(x-\xi_l)(x-\overline{\xi_l}),$$

where r_i are the real zeros, ξ complex, and $k+2l = n = \deg P$. Polynomial $(x-\xi)(x-\overline{\xi}) = x^2 - 2Re\xi + |\xi|^2 = x^2 - p_i x + q_i$ has real coefficients which satisfy $p_i^2 - 4q_i < 0$. This shows that:

Theorem 9. *A real polynomial $P(x)$ has a unique factorization (up to the order) of the form*

$$P(x) = (x-r_1)\cdots(x-r_k)(x^2-p_1x+q_1)\cdots(x^2-p_lx+q_l),$$

where r_i and p_j, q_j are real numbers with $p_i^2 < 4q_i$ and $k+2l = n$. □

It follows that a real polynomial of an odd degree always has an odd number of zeros (and at least one).

2 Zeros of Polynomials

In the first section we described some basic properties of polynomials. In this section we describe some further properties and at the end we prove that every complex polynomial actually has a root.

As we pointed out, in some cases the zeros of a given polynomial can be exactly determined. The case of polynomials of degree 2 has been known since the old age. The well-known formula gives the solutions of a quadratic equation $ax^2+bx+c = 0$ $(a \neq 0)$ in the form

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

When f has degree 3 or 4, the (fairly impractical) formulas describing the solutions were given by the Italian mathematicians Tartaglia and Ferrari in the 16-th century. We show Tartaglia's method of solving a cubic equation.

At first, substituting $x = y - a/3$ reduces the cubic equation $x^3+ax^2+bx+c = 0$ with real coefficients to

$$y^3+py+q = 0, \quad \text{where} \quad p = b - \frac{a^2}{3}, \quad q = c - \frac{ab}{3} + \frac{2a^3}{27}.$$

Putting $y = u+v$ transforms this equation into $u^3+v^3+(3uv+p)y+q = 0$. But, since u and v are variable, we are allowed to bind them by the condition $3uv+p = 0$. Thus the above equation becomes the system

$$uv = -\frac{p}{3}, \quad u^3+v^3 = -q$$

which is easily solved: u^3 and v^3 are the solutions of the quadratic equation $t^2+qt-\frac{p^3}{27} = 0$ and $uv = -p/3$ must be real. Thus we come to the solutions:

Theorem 10 (Cardano's formula). *The solutions of the equation $y^3+py+q = 0$ with $p,q \in \mathbb{R}$ are*

$$y_i = \varepsilon^j \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} + \varepsilon^{-j}\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}, \quad j = 0,1,2,$$

where ε is a primitive cubic root of unity. □

A polynomial $f(x) = a_nx^n+\cdots+a_1x+a_0$ is *symmetric* if $a_{n-i} = a_i$ for all i. If $\deg f = n$ is odd, then -1 is a zero of f and the polynomial $f(x)/(x+1)$ is symmetric. If $n = 2k$ is even, then

$$f(x)/x^k = a_0(x^k+x^{-k})+\cdots+a_{k-1}(x+x^{-1})+a_k$$

is a polynomial in $y = x+x^{-1}$, for so is each of the expressions x^i+x^{-i} (see problem 3 in section 7). In particular, $x^2+x^{-2} = y^2-2$, $x^3+x^{-3} = y^3-3y$, etc. This reduces the equation $f(x) = 0$ to an equation of degree $n/2$.

Problem 2. *Show that the polynomial* $f(x) = x^6 - 2x^5 + x^4 - 2x^3 + x^2 - 2x + 1$ *has exactly four zeros of modulus 1.*

Solution. Set $y = x + x^{-1}$. Then

$$\frac{f(x)}{x^3} = g(y) = y^3 - 2y^2 - 2y + 2.$$

Observe that x is of modulus 1 if and only if $x = \cos t + i \sin t$ for some t, in which case $y = 2\cos t$; conversely, $y = 2\cos t$ implies that $x = \cos t \pm i \sin t$. In other words, $|x| = 1$ if and only if y is real with $-2 \le y \le 2$, where to each such y correspond two values of x if $y \ne \pm 2$. Therefore it remains to show that $g(y)$ has exactly two real roots in the interval $(-2,2)$. To see this, it is enough to note that $g(-2) = -10$, $g(0) = 2$, $g(2) = -2$, and that therefore g has a zero in each of the intervals $(-2,0)$, $(0,2)$ and $(2,\infty)$. △

How are the roots of a polynomial related to its coefficients? Consider a monic polynomial

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = (x - x_1)(x - x_2)\cdots(x - x_n)$$

of degree $n > 0$. For example, comparing coefficients at x^{n-1} on both sides gives us $x_1 + x_2 + \cdots + x_n = -a_1$. Similarly, comparing the constant terms gives us $x_1 x_2 \cdots x_n = (-1)^n a_n$. The general relations are given by the Vieta formulas below.

Definition 1. Elementary symmetric polynomials *in* $x_1, \dots, x_n$ *are the polynomials* $\sigma_1, \sigma_2, \dots, \sigma_n$, *where*

$$\sigma_k = \sigma_k(x_1, x_2, \dots, x_n) = \sum x_{i_1} x_{i_2} \dots x_{i_k},$$

the sum being over all k-element subsets $\{i_1, \dots, i_k\}$ *of* $\{1,2,\dots,n\}$.

In particular, $\sigma_1 = x_1 + x_2 + \cdots + x_n$ and $\sigma_n = x_1 x_2 \cdots x_n$. Also, we usually set $\sigma_0 = 1$ and $\sigma_k = 0$ for $k > n$.

Theorem 11 (Vieta's formulas). *If* $\alpha_1, \alpha_2, \dots, \alpha_n$ *are the zeros of polynomial* $P(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$, *then* $a_k = (-1)^k \sigma_k(\alpha_1, \dots, \alpha_n)$ *for* $k = 1,2,\dots,n$.

Proof. Induction on n. The case $n = 1$ is trivial. Assume that $n > 1$ and write $P(x) = (x - x_n)Q(x)$, where $Q(x) = (x - x_1)\cdots(x - x_{n-1})$. Let us compute the coefficient a_k of $P(x)$ at x^k. Since the coefficients of $Q(x)$ at x^{k-1} and x^k are $a'_{k-1} = (-1)^{k-1}\sigma_{k-1}(x_1, \dots, x_{n-1})$ and $a'_k = (-1)^k \sigma_k(x_1, \dots, x_{n-1})$ respectively, we have

$$a_k = -x_n a'_{k-1} + a'_k = \sigma_k(x_1, \dots, x_n). \ \square$$

Example 4. *The roots* x_1, x_2, x_3 *of polynomial* $P(x) = x^3 - ax^2 + bx - c$ *satisfy* $a = x_1 + x_2 + x_3$, $b = x_1 x_2 + x_2 x_3 + x_3 x_1$ *and* $c = x_1 x_2 x_3$.

Problem 3. *Prove that not all zeros of a polynomial of the form* $x^n + 2nx^{n-1} + 2n^2 x^{n-2} + \cdots$ *can be real.*

Solution. Suppose that all its zeros $x_1, x_2, \dots, x_n$ are real. They satisfy

$$\sum_i x_i = -2n, \quad \sum_{i<j} x_i x_j = 2n^2.$$

However, by the mean inequality we have

$$\sum_{i<j} x_i x_j = \frac{1}{2}\left(\sum_i x_i\right)^2 - \frac{1}{2}\sum_i x_i^2 \le \frac{n-1}{2n}\left(\sum_i x_i\right)^2 = 2n(n-1),$$

a contradiction. △

Problem 4. *Find all polynomials of the form $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ with $a_j \in \{-1,1\}$ ($j = 0,1,\dots,n$), whose all roots are real.*

Solution. Let $x_1,\dots,x_n$ be the roots of the given polynomial. Then

$$x_1^2 + x_2^2 + \cdots + x_n^2 = (\sum_i x_i)^2 - 2(\sum_{i<j} x_ix_j) = a_{n-1}^2 - 2a_{n-2} \le 3;$$
$$x_1^2x_2^2\cdots x_n^2 = 1.$$

By the mean inequality, the second equality implies $x_1^2 + \cdots + x_n^2 \ge n$; hence $n \le 3$. The case $n = 3$ is only possible if $x_1,x_2,x_3 = \pm 1$. Now we can easily find all solutions: $x \pm 1$, $x^2 \pm x - 1$, $x^3 - x \pm (x^2 - 1)$. △

One contradiction is enough to show that not all zeros of a given polynomial are real. On the other hand, if the task is to show that all zeros of a polynomial *are* real, but not all are computable, the situation often gets more complicated.

Problem 5. *Show that all zeros of a polynomial $f(x) = x(x-2)(x-4)(x-6) + (x-1)(x-3)(x-5)(x-7)$ are real.*

Solution. Since $f(-\infty) = f(\infty) = +\infty$, $f(1) < 0$, $f(3) > 0$ and $f(5) < 0$, polynomial f has a real zero in each of the intervals $(-\infty,1)$, $(1,3)$, $(3,5)$, $(5,\infty)$, that is four in total. △

We now give the announced proof of the fact that every polynomial has a complex root. This fundamental theorem has many different proofs. The proof we present is, although more difficult than all the previous ones, still next to elementary. All imperfections in the proof are made on purpose.

Theorem 12 (The Fundamental Theorem of Algebra). *Every nonconstant complex polynomial $P(x)$ has a complex zero.*

Proof. Write $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$. Suppose that $P(0) = a_0 \ne 0$. For each $r > 0$, let C_r be the circle in the complex plane with the center at point 0 and radius r. Consider the continuous curve $\gamma_r = P(C_r) = \{P(x) \mid |x| = r\}$. The curve described by the monomial x^n, i.e. $\{x^n \mid x \in C_r\}$ rounds point 0 n times. If r is large enough, for example $r > 1 + |a_0| + \cdots + |a_{n-1}|$, we have $|x^n| > |a_{n-1}x^{n-1} + \cdots + a_0| = |P(x) - x^n|$, which means that the rest $P(x) - x^n$ in the expression of $P(x)$ can not "reach" point 0. Thus for such r the curve γ_r also rounds point 0 n times; hence, it contains point 0 in its interior.

For very small r the curve γ_r is close to point $P(0) = a_0$ and leaves point 0 in its exterior. Thus, there exists a minimum $r = r_0$ for which point 0 is *not* in the exterior of γ_r. Since the curve γ_r changes continuously as a function of r, it cannot jump over the point 0, so point 0 must lie on the curve γ_{r_0}. Therefore, there is a zero of polynomial $P(x)$ of modulus r_0. □

3 Polynomials with Integer Coefficients

Consider a polynomial $P(x) = a_nx^n + \cdots + a_1x + a_0$ with integer coefficients. The difference $P(x) - P(y)$ can be written in the form

$$a_n(x^n - y^n) + \cdots + a_2(x^2 - y^2) + a_1(x - y),$$

in which all summands are multiples of polynomial $x - y$. This leads to the simple though important arithmetic property of polynomials from $\mathbb{Z}[x]$:

Theorem 13. *If P is a polynomial with integer coefficients, then $P(a) - P(b)$ is divisible by $a - b$ for any distinct integers a and b.*

In particular, all integer roots of P divide $P(0)$. □

There is a similar statement about rational roots of polynomial $P(x) \in \mathbb{Z}[x]$.

Theorem 14. *If a rational number p/q ($p,q \in \mathbb{Z}$, $q \neq 0$, $nzd(p,q) = 1$) is a root of polynomial $P(x) = a_n x^n + \cdots + a_0$ with integer coefficients, then $p \mid a_0$ and $q \mid a_n$.*

Proof. We have

$$q^n P\left(\frac{p}{q}\right) = a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_0 q^n.$$

All summands but possibly the first are multiples of q, and all but possibly the last are multiples of p. Hence $q \mid a_n p^n$ and $p \mid a_0 q^n$ and the claim follows. □

Problem 6. *Polynomial $P(x) \in \mathbb{Z}[x]$ takes values ± 1 at three different integer points. Prove that it has no integer zeros.*

Solution. Suppose to the contrary, that a,b,c,d are integers with $P(a)$, $P(b), P(c) \in \{-1,1\}$ and $P(d) = 0$. Then by the previous statement the integers $a-d, b-d$ and $c-d$ all divide 1, a contradiction. △

Problem 7. *Let $P(x)$ be a polynomial with integer coefficients. Prove that if $P(P(\cdots P(x)\cdots)) = x$ for some integer x (where P is iterated n times), then $P(P(x)) = x$.*

Solution. Consider the sequence given by $x_0 = x$ and $x_{k+1} = P(x_k)$ for $k \geq 0$. Assume $x_k = x_0$. We know that

$$d_i = x_{i+1} - x_i \mid P(x_{i+1}) - P(x_i) = x_{i+2} - x_{i+1} = d_{i+1}$$

for all i, which together with $d_k = d_0$ implies $|d_0| = |d_1| = \cdots = |d_k|$.

Suppose that $d_1 = d_0 = d \neq 0$. Then $d_2 = d$ (otherwise $x_3 = x_1$ and x_0 will never occur in the sequence again). Similarly, $d_3 = d$ etc, and hence $x_k = x_0 + kd \neq x_0$ for all k, a contradiction. It follows that $d_1 = -d_0$, so $x_2 = x_0$. △

Note that a polynomial that takes integer values at all integer points does not necessarily have integer coefficients, as seen on the polynomial $\frac{x(x-1)}{2}$.

Theorem 15. *If the value of the polynomial $P(x)$ is integral for every integer x, then there exist integers $c_0, \ldots, c_n$ such that*

$$P(x) = c_n \binom{x}{n} + c_{n-1} \binom{x}{n-1} + \cdots + c_0 \binom{x}{0}.$$

The converse is true, also.

Proof. We use induction on n. The case $n = 1$ is trivial; Now assume that $n > 1$. Polynomial $Q(x) = P(x+1) - P(x)$ is of degree $n-1$ and takes integer values at all integer points, so by the inductive hypothesis there exist $a_0, \ldots, a_{n-1} \in \mathbb{Z}$ such that

$$Q(x) = a_{n-1} \binom{x}{n-1} + \cdots + a_0 \binom{x}{0}.$$

For every integer $x > 0$ we have $P(x) = P(0) + Q(0) + Q(1) + \cdots + Q(x-1)$. Using the identity $\binom{0}{k} + \binom{1}{k} + \cdots + \binom{x-1}{k} = \binom{x}{k+1}$ for every integer k we obtain the desired representation of $P(x)$:

$$P(x) = a_{n-1} \binom{x}{n} + \cdots + a_0 \binom{x}{1} + P(0). \ \square$$

Problem 8. *Suppose that a natural number m and a real polynomial $R(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ are such that $R(x)$ is an integer divisible by m whenever x is an integer. Prove that $n!a_n$ is divisible by m.*

Solution. Apply the previous theorem on polynomial $\frac{1}{m} R(x)$ (with the same notation). The leading coefficient of this polynomial equals $c_n + nc_{n-1} + \cdots + n!c_0$, and the statement follows immediately. △

4 Irreducibility

Polynomial $P(x)$ with integer coefficients is said to be *irreducible* over $\mathbb{Z}[x]$ if it cannot be written as a product of two nonconstant polynomials with integer coefficients.

Example 5. *Every quadratic or cubic polynomial with no rational roots is irreducible over $\mathbb{Z}$. Such are e.g. x^2-x-1 and $2x^3-4x+1$.*

One analogously defines (ir)reducibility over the sets of polynomials with e.g. rational, real or complex coefficients. However, of the mentioned, only reducibility over $\mathbb{Z}[x]$ is of interest. Gauss' Lemma below claims that the reducibility over $\mathbb{Q}[x]$ is equivalent to the reducibility over $\mathbb{Z}[x]$. In addition, we have already shown that a real polynomial is always reducible into linear and quadratic factors over $\mathbb{R}[x]$, while a complex polynomial is always reducible into linear factors over $\mathbb{C}[x]$.

Theorem 16 (Gauss' Lema). *If a polynomial $P(x)$ with integer coefficients is reducible over $\mathbb{Q}[x]$, then it is reducible over $\mathbb{Z}[x]$, also.*

Proof. Suppose that $P(x) = a_nx^n + \cdots + a_0 = Q(x)R(x) \in \mathbb{Z}[x]$, where $Q(x)$ and $R(x)$ nonconstant polynomials with rational coefficients. Let q and r be the smallest natural numbers such that the polynomials $qQ(x) = q_kx^k + \cdots + q_0$ and $rR(x) = r_mx^m + \cdots + r_0$ have integer coefficients. Then $qrP(x) = qQ(x) \cdot rR(x)$ is a factorization of the polynomial $qrP(x)$ into two polynomials from $\mathbb{Z}[x]$. Based on this, we shall construct such a factorization for $P(x)$.

Let p be an arbitrary prime divisor of q. All coefficients of $P(x)$ are divisible by p. Let i be such that $p \mid q_0, q_1, \dots, q_{i-1}$, but $p \nmid q_i$. We have $p \mid a_i = q_0r_i + \cdots + q_ir_0 \equiv q_ir_0 \pmod p$, which implies that $p \mid r_0$. Furthermore, $p \mid a_{i+1} = q_0r_{i+1} + \cdots + q_ir_1 + q_{i+1}r_0 \equiv q_ir_1 \pmod p$, so $p \mid r_1$. Continuing in this way, we deduce that $p \mid r_j$ for all j. Hence $rR(x)/p$ has integer coefficients. We have thus obtained a factorization of $\frac{rq}{p}P(x)$ into two polynomials from $\mathbb{Z}[x]$. Continuing this procedure and taking other values for p we shall eventually end up with a factorization of $P(x)$ itself. □

From now on, unless otherwise specified, by "irreducibility" we mean irreducibility over $\mathbb{Z}[x]$.

Problem 9. *If $a_1, a_2, \dots, a_n$ are integers, prove that the polynomial $P(x) = (x-a_1)(x-a_2)\cdots(x-a_n)-1$ is irreducible.*

Solution. Suppose that $P(x) = Q(x)R(x)$ for some nonconstant polynomials $Q, R \in \mathbb{Z}[x]$. Since $Q(a_i)R(a_i) = -1$ for $i = 1, \dots, n$, we have $Q(a_i) = 1$ and $R(a_i) = -1$ or $Q(a_i) = -1$ and $R(a_i) = 1$; either way, we have $Q(a_i) + R(a_i) = 0$. It follows that the polynomial $Q(x) + R(x)$ (which is obviously nonzero) has n zeros $a_1, \dots, a_n$ which is impossible for its degree is less than n. △

Theorem 17 (Extended Eisenstein's Criterion). *Let $P(x) = a_nx^n + \cdots + a_1x + a_0$ be a polynomial with integer coefficients. If there exist a prime number p and an integer $k \in \{0, 1, \dots, n-1\}$ such that*

$$p \mid a_0, a_1, \dots, a_k, \;\; p \nmid a_{k+1} \;\text{ and }\; p^2 \nmid a_0,$$

then $P(x)$ has an irreducible factor of a degree greater than k.

In particular, if p can be taken so that $k = n-1$, then $P(x)$ is irreducible.

Proof. Like in the proof of Gauss's lemma, suppose that $P(x) = Q(x)R(x)$, where $Q(x) = q_kx^k + \cdots + q_0$ and $R(x) = r_mx^m + \cdots + r_0$ are polynomials from $\mathbb{Z}[x]$. Since $a_0 = q_0r_0$ is divisible by p and not by p^2, exactly one of q_0, r_0 is a multiple of p. Assume that $p \mid q_0$ and $p \nmid r_0$. Further, $p \mid a_1 = q_0r_1 + q_1r_0$, implying that $p \mid q_1r_0$, i.e. $p \mid q_1$, and so on. We conclude that all coefficients $q_0, q_1, \dots, q_k$ are divisible by p, but $p \nmid q_{k+1}$. It follows that $\deg Q \geq k+1$. □

Problem 10. *Given an integer $n > 1$, consider the polynomial $f(x) = x^n + 5x^{n-1} + 3$. Prove that there are no nonconstant polynomials $g(x), h(x)$ with integer coefficients such that $f(x) = g(x)h(x)$. (IMO93-1)*

Solution. By the (extended) Eisenstein criterion, f has an irreducible factor of degree at least $n-1$. Since f has no integer zeros, it must be irreducible. △

Problem 11. *If p is a prime number, prove that the polynomial $\Phi_p(x) = x^{p-1} + \cdots + x + 1$ is irreducible.*

Solution. Instead of $\Phi_p(x)$, we shall consider $\Phi_p(x+1)$ and show that it is irreducible, which will clearly imply that so is Φ_p. We have

$$\Phi_p(x+1) = \frac{(x+1)^p - 1}{x} = x^{p-1} + \binom{p}{p-1}x^{p-2} + \cdots + \binom{p}{2}x + p.$$

This polynomial satisfies all the assumptions of Eisenstein's criterion, based on which it is irreducible. △

In investigating reducibility of a polynomial, it can be useful to investigate its zeros and their modules. The following problems provide us an illustration.

Problem 12. *Prove that the polynomial $P(x) = x^n + 4$ is irreducible over $\mathbb{Z}[x]$ if and only if n is a multiple of 4.*

Solution. All zeros of polynomial P have the modulus equal to $2^{2/n}$. If Q and R are polynomials from $\mathbb{Z}[x]$ and $\deg Q = k$, then $|Q(0)|$ is the product of the modules of the zeros of Q and equals $2^{2k/n}$; since this should be an integer, we deduce that $n = 2k$.

If k is odd, polynomial Q has a real zero, which is impossible since $P(x)$ has none. Therefore, $2 \mid k$ and $4 \mid n$. △

If the zeros cannot be exactly determined, one should find a good enough bound. Estimating complex zeros of a polynomial is not always simple. Our main tool is the triangle inequality for complex numbers:

$$|x| - |y| \le |x+y| \le |x| + |y|.$$

Consider a polynomial $P(x) = a_n x^n + a_{n-k} xn - k + \cdots + a_1 x + a_0$ with complex coefficients ($a_n \neq 0$). Let α be its zero. If M is a real number such that $|a_i| < M|a_n|$ for all i, it holds that

$$0 = |P(\alpha)| \ge |a_n||\alpha|^n - M|a_n|(|\alpha|^{n-k} + \cdots + |\alpha| + 1) > |a_n||\alpha|^n \left(1 - \frac{M}{|\alpha|^{k-1}(|\alpha|-1)}\right),$$

which yields $|\alpha|^{k-1}(|\alpha|-1) < M$. We thus come to the following estimate:

Theorem 18. *Let $P(x) = a_n x^n + \cdots + a_0$ be a complex polynomial with $a_n \neq 0$ and $M = \max_{0 \le k < n} \left|\frac{a_k}{a_n}\right|$. If $a_{n-1} = \cdots = a_{n-k+1} = 0$, then all roots of the polynomial P are less than $1 + \sqrt[k]{M}$ in modulus.*

In particular, for $k = 1$, each zero of $P(x)$ is of modulus less than $M + 1$. □

Problem 13. *If $\overline{a_n \ldots a_1 a_0}$ is a decimal representation of a prime number and $a_n > 1$, prove that the polynomial $P(x) = a_n x^n + \cdots + a_1 x + a_0$ is irreducible. (*BMO 1989.2*)*

Solution. Suppose that Q and R are nonconstant polynomials from $\mathbb{Z}[x]$ with $Q(x)R(x) = P(x)$. Let $x_1, \ldots, x_k$ be the zeros of Q and $x_{k+1}, \ldots, x_n$ be the zeros of R. The condition of the problem means that $P(10) = Q(10)R(10)$ is a prime, so we can assume w.l.o.g. that

$$|Q(10)| = (10 - x_1)(10 - x_2)\cdots(10 - x_k) = 1.$$

On the other hand, by the estimate in 18, each zero x_i has a modulus less than $1 + 9/2 = 11/2 < 9$; hence $|10 - x_i| > 1$ for all i, contradicting the above inequality. △

Problem 14. *Let $p > 2$ be a prime number and $P(x) = x^p - x + p$.*

1. *Prove that all zeros of polynomial P are less than $p^{\frac{1}{p-1}}$ in modulus.*

2. *Prove that the polynomial $P(x)$ is irreducible.*

Solution.

1. Let y be a zero of P. Then $|y|^p - |y| \leq |y^p - y| = p$. If we assume that $|y| \geq p^{\frac{1}{p-1}}$, we obtain
$$|y|^p - |y| \geq (p-1)p^{\frac{1}{p-1}} > p,$$
a contradiction. Here we used the inequality $p^{\frac{1}{p-1}} > \frac{p}{p-1}$ which follows for example from the binomial expansion of $p^{p-1} = ((p-1)+1)^{p-1}$.

2. Suppose that $P(x)$ is the product of two nonconstant polynomials $Q(x)$ and $R(x)$ with integer coefficients. One of these two polynomials, say Q, has the constant term equal to $\pm p$. On the other hand, the zeros $x_1, \dots, x_k$ of Q satisfy $|x_1|, \dots, |x_k| < p^{\frac{1}{p-1}}$ by part (a), and $x_1 \cdots x_k = \pm p$, so we conclude that $k \geq p$, which is impossible. $\triangle$

5 Interpolating polynomials

A polynomial of n-th degree is uniquely determined, given its values at $n+1$ points. So, suppose that P is an n-th degree polynomial and that $P(x_i) = y_i$ in different points $x_0, x_1, \dots, x_n$. There exist unique polynomials $E_0, E_1, \dots, E_n$ of n-th degree such that $E_i(x_i) = 1$ and $E_i(x_j) = 0$ for $j \neq i$. Then the polynomial
$$P(x) = y_0E_0(x) + y_1E_1(x) + \cdots + y_nE_n(x)$$
has the desired properties: indeed, $P(x_i) = \sum_j y_jE_j(x_i) = y_iE_i(x_i) = y_i$. It remains to find the polynomials $E_0, \dots, E_n$. A polynomial that vanishes at the n points x_j, $j \neq i$, is divisible by $\prod_{j\neq i}(x - x_j)$, from which we easily obtain $E_i(x) = \prod_{j\neq i} \frac{(x-x_j)}{(x_i-x_j)}$. This shows that:

Theorem 19 (Newton's interpolating polynomial). *For given numbers $y_0, \dots, y_n$ and distinct x_0, $\dots$, x_n there is a unique polynomial $P(x)$ of n-th degree such that $P(x_i) = y_i$ for $i = 0, 1, \dots, n$. This polynomial is given by the formula*
$$P(x) = \sum_{i=0}^{n} y_i \prod_{j\neq i} \frac{(x - x_j)}{(x_i - x_j)}. \quad \square$$

Example 6. *Find the cubic polynomial Q such that $Q(i) = 2^i$ for $i = 0, 1, 2, 3$.*

Solution. $Q(x) = \frac{(x-1)(x-2)(x-3)}{-6} + \frac{2x(x-2)(x-3)}{2} + \frac{4x(x-1)(x-3)}{-2} + \frac{8x(x-1)(x-2)}{6} = \frac{x^3+5x+6}{6}$. $\triangle$

In order to compute the value of a polynomial given in this way in some point, sometimes we do not need to determine its Newton's polynomial. In fact, Newton's polynomial has an unpleasant property of giving the answer in a complicated form.

Example 7. *If the polynomial P of n-th degree takes the value 1 in points $0, 2, 4, \dots, 2n$, compute $P(-1)$.*

Solution. $P(x)$ is of course identically equal to 1, so $P(-1) = 1$. But if we apply the Newton polynomial, here is what we get:
$$P(1) = \sum_{i=0}^{n} \prod_{j\neq i} \frac{1-2i}{(2j-2i)} = \sum_{i=0}^{n} \prod_{j\neq i} \frac{-1-2j}{(2i-2j)} = \frac{(2n+1)!!}{2^n} \sum_{i=1}^{n+1} \frac{(-1)^{n-i}}{(2i+1)i!(n-i)!}. \quad \triangle$$

Instead, it is often useful to consider the *finite difference* of polynomial P, defined by $P^{[1]}(x) = P(x+1) - P(x)$, which has the degree by 1 less than that of P. Further, we define the k-th finite difference, $P^{[k]} = (P^{[k-1]})^{[1]}$, which is of degree $n-k$ (where $\deg P = n$). A simple induction gives a general formula

$$P^{[k]} = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} P(x+i).$$

In particular, $P^{[n]}$ is constant and $P^{[n+1]} = 0$, which leads to

Theorem 20. $P(x+n+1) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n+1}{i} P(x+i)$. $\square$

Problem 15. *Polynomial P of degree n satisfies $P(i) = \binom{n+1}{i}^{-1}$ for $i = 0, 1, \dots, n$. Evaluate $P(n+1)$.*

Solution. We have

$$0 = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} P(i) = (-1)^{n+1} P(n+1) + \begin{cases} 1, & 2 \mid n; \\ 0, & 2 \nmid n. \end{cases}$$

It follows that $P(n+1) = \begin{cases} 1, & 2 \mid n; \\ 0, & 2 \nmid n. \end{cases}$ $\triangle$

Problem 16. *If $P(x)$ is a polynomial of an even degree n with $P(0) = 1$ and $P(i) = 2^{i-1}$ for $i = 1, \dots, n$, prove that $P(n+2) = 2P(n+1) - 1$.*

Solution. We observe that $P^{[1]}(0) = 0$ i $P^{[1]}(i) = 2^{i-1}$ for $i = 1, \dots, n-1$; furthermore, $P^{[2]}(0) = 1$ i $P^{[2]}(i) = 2^{i-1}$ for $i = 1, \dots, n-2$, etc. In general, it is easily seen that $P^{[k]}(i) = 2^{i-1}$ for $i = 1, \dots, n-k$, and $P^{[k]}(0)$ is 0 for k odd and 1 for k even. Now

$$P(n+1) = P(n) + P^{[1]}(n) = \cdots = P(n) + P^{[1]}(n-1) + \cdots + P^{[n]}(0) = \begin{cases} 2^n, & 2 \mid n; \\ 2^n - 1, & 2 \nmid n. \end{cases}$$

Similarly, $P(n+2) = 2^{2n+1} - 1$. $\triangle$

6 Applications of Calculus

The derivative of a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is given by

$$P'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1.$$

The inverse operation, the indefinite integral, is given by

$$\int P(x)dx = \frac{a_n}{n+1} x^{n+1} + \frac{a_{n-1}}{n} x^n + \cdots + a_0 x + C.$$

If the polynomial P is not given by its coefficients but rather by its canonical factorization, as $P(x) = (x-x_1)^{k_1} \cdots (x-x_n)^{k_n}$, a more suitable expression for the derivative is obtained by using the logarithmic derivative rule or product rule:

$$P'(x) = P(x) \left(\frac{k_1}{x-x_1} + \cdots + \frac{k_n}{x-x_n} \right).$$

A similar formula can be obtained for the second derivative.

Problem 17. *Suppose that real numbers* $0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1$ *satisfy*

$$\sum_{j=0, j\neq i}^{n+1} \frac{1}{x_i - x_j} = 0 \quad za\ i = 1,2,\dots,n. \tag{1}$$

Prove that $x_{n+1-i} = 1 - x_i$ *for* $i = 1,2,\dots,n$.

Solution. Let $P(x) = (x-x_0)(x-x_1)\cdots(x-x_n)(x-x_{n+1})$. We have

$$P'(x) = \sum_{j=0}^{n+1} \frac{P(x)}{x-x_j} \quad \text{i} \quad P''(x) = \sum_{j=0}^{n+1}\sum_{k\neq j} \frac{P(x)}{(x-x_j)(x-x_k)}.$$

Therefore

$$P''(x_i) = 2P'(x_i)\sum_{j\neq i} \frac{1}{(x_i - x_j)}$$

for $i = 0,1,\dots,n+1$. Thus the condition of the problem is equivalent to $P''(x_i) = 0$ for $i = 1,2,\dots,n$. Therefore

$$x(x-1)P''(x) = (n+2)(n+1)P(x).$$

It is easy to see that there is a unique monic polynomial of degree $n+2$ satisfying the above differential equation. On the other hand, the monic polynomial $Q(x) = (-1)^n P(1-x)$ satisfies the same equation and has degree $n+2$, so we must have $(-1)^n P(1-x) = P(x)$, which implies the statement. $\triangle$

What makes derivatives of polynomials especially suitable is their property of preserving multiple zeros.

Theorem 21. *If* $(x-\alpha)^k \mid P(x)$*, then* $(x-\alpha)^{k-1} \mid P'(x)$.

Proof. If $P(x) = (x-\alpha)^k Q(x)$, then $P'(x) = (x-\alpha)^k Q'(x) + k(x-\alpha)^{k-1}Q(x)$. $\square$

Problem 18. *Determine a real polynomial* $P(x)$ *of degree at most 5 which leaves remainders* -1 *and 1 upon division by* $(x-1)^3$ *and* $(x+1)^3$*, respectively.*

Solution. If $P(x)+1$ has a triple zero at point 1, then its derivative $P'(x)$ has a double zero at that point. Similarly, $P'(x)$ has a double zero at point -1 too. It follows that $P'(x)$ is divisible by the polynomial $(x-1)^2(x+1)^2$. Since $P'(x)$ is of degree at most 4, it follows that

$$P'(x) = c(x-1)^2(x+1)^2 = c(x^4 - 2x^2 + 1)$$

for some constant c. Now $P(x) = c(\frac{1}{5}x^5 - \frac{2}{3}x^3 + x) + d$ for some real numbers c and d. The conditions $P(-1) = 1$ and $P(1) = -1$ now give us $c = -15/8$, $d = 0$ and

$$P(x) = -\frac{3}{8}x^5 + \frac{5}{4}x^3 - \frac{15}{8}x. \ \triangle$$

Problem 19. *For polynomials* $P(x)$ *and* $Q(x)$ *and an arbitrary* $k \in \mathbb{C}$*, denote*

$$P_k = \{z \in \mathbb{C} \mid P(z) = k\} \quad and \quad Q_k = \{z \in \mathbb{C} \mid Q(z) = k\}.$$

Prove that $P_0 = Q_0$ *and* $P_1 = Q_1$ *imply that* $P(x) = Q(x)$.

Solution. Let us assume w.l.o.g. that $n = \deg P \geq \deg Q$. Let $P_0 = \{z_1, z_2, \dots, z_k\}$ and $P_1 = \{z_{k+1}, z_{k+2}, \dots, z_{k+m}\}$. Polynomials P and Q coincide at $k+m$ points $z_1, z_2, \dots, z_{k+m}$. The result will follow if we show that $k+m > n$.

We have

$$P(x) = (x-z_1)^{\alpha_1} \cdots (x-z_k)^{\alpha_k} = (x-z_{k+1})^{\alpha_{k+1}} \cdots (x-z_{k+m})^{\alpha_{k+m}} + 1$$

for some natural numbers $\alpha_1, \dots, \alpha_{k+m}$. Let us consider $P'(x)$. We know that it is divisible by $(x-z_i)^{\alpha_i - 1}$ for $i = 1, 2, \dots, k+m$; hence,

$$\prod_{i=1}^{k+m} (x-z_i)^{\alpha_i - 1} \mid P'(x).$$

Therefore, $2n-k-m = \deg \prod_{i=1}^{k+m} (x-z_i)^{\alpha_i - 1} \leq \deg P' = n-1$, i.e. $k+m \geq n+1$, as desired. △

Even if P has no multiple zeros, certain relations between zeros of P and P' still hold. For example, the following statement holds for all differentiable functions.

Theorem 22 (Rolle's Theorem). *Between every two zeros of a polynomial $P(x)$ there is a zero of $P'(x)$.*

Corollary. If all zeros of $P(x)$ are real, then so are all zeros of $P'(x)$.)
Proof. Let $a < b$ be two zeros of polynomial P. Assume w.l.o.g. that $P'(a) > 0$ and consider the point c in the interval $[a, b]$ in which P attains a local maximum (such a point exists since the interval $[a, b]$ is compact). We know that $P(x) = P(c) + (x-c)[P'(c) + o(1)]$. If for example $P'(c) > 0$ (the case $P'(c) < 0$ leads to a similar contradiction), then $P(x) > P(c)$ would hold in a small neighborhood of c, a contradiction. It is only possible that $P'(c) = 0$, so c is a root of $P'(x)$ between a and b. □

7 Symmetric polynomials

A symmetric polynomial in variables $x_1, \dots, x_n$ is every polynomial that is not varied by permuting the indices of the variables. For instance, polynomial x_1^2 is symmetric as a polynomial in x_1 (no wonder), but is not symmetric as a polynomial in x_1, x_2 as changing places of the indices 1 and 2 changes it to the polynomial x_2^2.

Definition 2. *The polynomial $P(x_1, x_2, \dots, x_n)$ is* symmetric *if, for every permutation π of $\{1, 2, \dots, n\}$, $P(x_1, x_2, \dots, x_n) \equiv P(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)})$.*

An obvious property of a symmetric polynomial is that its coefficients at two terms of the forms $x_1^{i_1} \cdots x_n^{i_n}$ and $x_1^{j_1} \cdots x_n^{j_n}$, where $(j_1, \dots, j_n)$ is a permutation $(i_1, \dots, i_n)$, always coincide. For example, if the expansion of a symmetric polynomial in x, y, z contains the terms $x^2 y$, then it also contains $x^2 z, xy^2$, etc, with the same coefficient.

Thus, the polynomials σ_k $(1 \leq k \leq n)$ introduced in section 2 are symmetric. Also symmetric is e.g. the polynomial $x_1^2 + x_2^2$.

A symmetric polynomial is said to be *homogenous* if all its terms are of the same degree. Equivalently, polynomial T is homogenous of degree d if $T(tx_1, \dots, tx_n) = t^d T(x_1, \dots, x_n)$ holds for all x and t. For instance, $x_1^2 + x_2^2$ is homogenous of degree $d = 2$, but $x_1^2 + x_2^2 + 1$, although symmetric, is not homogenous.

Every symmetric polynomial in $x_1, \dots, x_n$ can be written as a sum of homogenous polynomials. Moreover, it can also be represented as a linear combination of certain "bricks". These bricks are the polynomials

$$T_a = \sum x_1^{a_{i_1}} \cdots x_n^{a_{i_n}} \qquad (*)$$

for each n-tuple $a = (a_1, \dots, a_n)$ of nonnegative integers with $a_1 \geq \cdots \geq a_n$, where the summation goes over all permutations $(i_1, \dots, i_n)$ of the indices $1, \dots, n$. In the expression for T_a the same summand can occur more than once, so we define S_a as the sum of the *different* terms in $(*)$. The polynomial T_a is always an integral multiple of S_a. For instance,

$$T_{(2,2,0)} = 2(x_1^2 x_2^2 + x_2^2 x_3^2 + x_3^2 x_1^2) = 2S_{(2,2,0)}.$$

All the n-tuples a of degree $d = a_1 + \cdots + a_n$ can be ordered in a lexicographic order so that

$$a > a' \quad \text{if} \quad s_1 = s'_1, \dots, s_k = s'_k \text{ and } s_{k+1} > s'_{k+1} \quad \text{for some } k \geq 1,$$

where $s_i = a_1 + \cdots + a_i$. In this ordering, the least n-tuple is $m = (x+1, \dots, x+1, x, \dots, x)$, where $x = [d/n]$ and $x+1$ occurs $d - n[d/n]$ times.

The polynomials T_a can be multiplies according to the following simple formula:

Theorem 23. *If $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are n-tuples of nonnegative integers, it holds that*

$$T_a \cdot T_b = \sum_{\pi} T_{a+\pi(b)},$$

where the sum goes over all permutations $\pi(b)$ of the n-tuple b. (We define $(x_i)_{i=1}^n + (y_i)_{i=1}^n = (x_i + y_i)_{i=1}^n$.)

Proof. It suffices to observe that

$$x_1^{\pi_1(b)} \cdots x_n^{\pi_n(b)} T_a = \sum x_{i_1}^{a_1+\pi_{i_1}(b)} \cdots x_{i_n}^{a_n+\pi_{i_n}(b)},$$

and to sum up over all permutations π. □

There are infinitely many mentioned bricks, and these are obviously not mutually independent. We need simpler elements which are independent and using which one can express every symmetric polynomial by basic operations. It turns out that these atoms are $\sigma_1, \dots, \sigma_n$.

Example 8. *The following polynomials in x, y, z can be written in terms of $\sigma_1, \sigma_2, \sigma_3$:*
$xy + yz + zx + x + y + z = \sigma_2 + \sigma_1$;
$x^2y + x^2z + y^2x + y^2z + z^2x + z^2y = \sigma_1\sigma_2 - 3\sigma_3$;
$x^2y^2 + y^2z^2 + z^2x^2 = \sigma_2^2 - 2\sigma_1\sigma_3$.

Theorem 24. *Every symmetric polynomial in $x_1, \dots, x_n$ can be represented in the form of a polynomial in $\sigma_1, \dots, \sigma_n$. Moreover, a symmetric polynomial with integer coefficients is also a polynomial in $\sigma_1, \dots, \sigma_n$ with integer coefficients.*

Proof. It is enough to prove the statement for the polynomials S_a of degree d (for each d). Assuming that it holds for the degrees less than d, we use induction on n-tuples a. The statement is true for the smallest n-tuple m: Indeed, $S_m = \sigma_n^q \sigma_r$, where $d = nq + r$, $0 \leq r < n$. Now suppose that the statement is true for all S_b with $b < a$; we show that it also holds for S_a.

Suppose that $a = (a_1, \dots, a_n)$ with $a_1 = \cdots = a_k > a_{k+1}$ ($k \geq 1$). Consider the polynomial $S_a - \sigma_k S_{a'}$, where $a' = (a_1 - 1, \dots, a_k - 1, a_{k+1}, \dots, a_n)$. According to theorem 23 it is easy to see that this polynomial is of the form $\sum_{b<a} c_b S_b$, where c_b are integers, and is therefore by the inductive hypothesis representable in the form of a polynomial in σ_i with integer coefficients. □

The proof of the previous theorem also gives us an algorithm for expressing each symmetric polynomial in terms of the σ_i. Nevertheless, for some particular symmetric polynomials there are simpler formulas.

Theorem 25 (Newton's Theorem on Symmetric Polynomials). *If we denote $s_k = x_1^k + x_2^k + \cdots + x_n^k$, then:*

$$\begin{aligned} k\sigma_k &= s_1\sigma_{k-1} - s_2\sigma_{k-2} + \cdots + (-1)^k s_{k-1}\sigma_1 + (-1)^{k+1} s_k; \\ s_m &= \sigma_1 s_{m-1} - \sigma_2 s_{m-2} + \cdots + (-1)^{n-1}\sigma_n s_{m-n} \quad za\ m \geq n. \end{aligned}$$

(All the polynomials are in n variables.)

Proof. Direct, for example by using the formula 23. □

Problem 20. *Suppose that complex numbers $x_1, x_2, \dots, x_k$ satisfy*

$$x_1^j + x_2^j + \dots + x_k^j = n, \qquad \text{for } j = 1, 2, \dots, k,$$

where n, k are given positive integers. Prove that

$$(x - x_1)(x - x_2)\dots(x - x_k) = x^k - \binom{n}{1}x^{k-1} + \binom{n}{2}x^{k-2} - \dots + (-1)^k\binom{n}{k}.$$

Solution. We are given $s_k = n$ for $k = 1, \dots, n$. The Newton's theorem gives us $\sigma_1 = n$, $\sigma_2 = \frac{1}{2}(n\sigma_1 - n) = \binom{n}{2}$, $\sigma_3 = \frac{1}{3}(n\sigma_2 - n\sigma_1 + n) = \binom{n}{3}$, etc. We prove by induction on k that $\sigma_k = \binom{n}{k}$. If this holds for $1, \dots, k-1$, we have

$$\sigma_k = \frac{n}{k}\left[\binom{n}{k-1} - \binom{n}{k-2} + \binom{n}{k-3} - \dots\right].$$

Since $\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}$, the above equality telescopes to $\sigma_k = \frac{n}{k}\binom{n-1}{k-1}$, which is exactly equal to $\binom{n}{k}$. △

8 Problems

1. A monic polynomial $f(x)$ of fourth degree satisfies $f(1) = 10$, $f(2) = 20$ and $f(3) = 30$. Determine $f(12) + f(-8)$.

2. Consider complex polynomials $P(x) = x^n + a_1x^{n-1} + \dots + a_n$ with the zeros $x_1, \dots, x_n$, and $Q(x) = x^n + b_1x^{n-1} + \dots + b_n$ with the zeros $x_1^2, \dots, x_n^2$. Prove that if $a_1 + a_3 + a_5 + \cdots$ and $a_2 + a_4 + a_6 + \cdots$ are real numbers, then $b_1 + b_2 + \dots + b_n$ is also real.

3. If a polynomial P with real coefficients satisfies for all x

   $$P(\cos x) = P(\sin x),$$

   show that there exists a polynomial Q such that $P(x) = Q(x^4 - x^2)$ for each x.

4. (a) Prove that for each $n \in \mathbb{N}$ there is a polynomial T_n with integer coefficients and the leading coefficient 2^{n-1} such that $T_n(\cos x) = \cos nx$ for all x.

   (b) Prove that the polynomials T_n satisfy $T_{m+n} + T_{m-n} = 2T_mT_n$ for all $m, n \in \mathbb{N}$, $m \geq n$.

   (c) Prove that the polynomial U_n given by $U_n(2x) = 2T_n(x)$ also has integer coefficients and satisfies $U_n(x + x^{-1}) = x^n + x^{-n}$.

   The polynomials $T_n(x)$ are known as the *Chebyshev polynomials*.

5. Prove that if $\cos\frac{p}{q}\pi = a$ is a rational number for some $p, q \in \mathbb{Z}$, then $a \in \{0, \pm\frac{1}{2}, \pm 1\}$.

6. Prove that the maximum in absolute value of any monic real polynomial of n-th degree on $[-1, 1]$ is not less than $\frac{1}{2^{n-1}}$.

7. The polynomial P of n-th degree is such that, for each $i = 0, 1, \dots, n$, $P(i)$ equals the remainder of i modulo 2. Evaluate $P(n+1)$.

8. A polynomial $P(x)$ of n-th degree satisfies $P(i) = \frac{1}{i}$ for $i = 1, 2, \dots, n+1$. Find $P(n+2)$.

9. Let $P(x)$ be a real polynomial.

   (a) If $P(x) \geq 0$ for all x, show that there exist real polynomials $A(x)$ and $B(x)$ such that $P(x) = A(x)^2 + B(x)^2$.

(b) If $P(x) \geq 0$ for all $x \geq 0$, show that there exist real polynomials $A(x)$ and $B(x)$ such that $P(x) = A(x)^2 + xB(x)^2$.

10. Prove that if the equation $Q(x) = ax^2 + (c-b)x + (e-d) = 0$ has real roots greater than 1, where $a, b, c, d, e \in \mathbb{R}$, then the equation $P(x) = ax^4 + bx^3 + cx^2 + dx + e = 0$ has at least one real root.

11. A monic polynomial P with real coefficients satisfies $|P(i)| < 1$. Prove that there is a root $z = a + bi$ of P such that $(a^2 + b^2 + 1)^2 < 4b^2 + 1$.

12. For what real values of a does there exist a rational function $f(x)$ that satisfies $f(x^2) = f(x)^2 - a$? (A rational function is a quotient of two polynomials.)

13. Find all polynomials P satisfying $P(x^2 + 1) = P(x)^2 + 1$ for all x.

14. Find all P for which $P(x)^2 - 2 = 2P(2x^2 - 1)$.

15. If the polynomials P and Q each have a real root and

$$P(1 + x + Q(x)^2) = Q(1 + x + P(x)^2),$$

prove that $P \equiv Q$.

16. Find all polynomials $P(x)$ with real coefficients satisfying the equality

$$P(a-b) + P(b-c) + P(c-a) = 2P(a+b+c)$$

for all triples (a, b, c) of real numbers such that $ab + bc + ca = 0$. *(IMO04-2)*

17. A sequence of integers $(a_n)_{n=1}^{\infty}$ has the property that $m - n \mid a_m - a_n$ for any distinct $m, n \in \mathbb{N}$. Suppose that there is a polynomial $P(x)$ such that $|a_n| < P(n)$ for all n. Show that there exists a polynomial $Q(x)$ such that $a_n = Q(n)$ for all n.

18. Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a natural number. Consider the polynomial $Q(x) = P(P(\dots P(P(x))\dots))$, where P is applied k times. Prove that there exist at most n integers t such that $Q(t) = t$. *(IMO06-5)*

19. If P and Q are monic polynomials such that $P(P(x)) = Q(Q(x))$, prove that $P \equiv Q$.

20. Let m, n and a be natural numbers and $p < a - 1$ a prime number. Prove that the polynomial $f(x) = x^m(x-a)^n + p$ is irreducible.

21. Prove that the polynomial $F(x) = (x^2 + x)^{2^n} + 1$ is irreducible for all $n \in \mathbb{N}$.

22. A polynomial $P(x)$ has the property that for every $y \in \mathbb{Q}$ there exists $x \in \mathbb{Q}$ such that $P(x) = y$. Prove that P is a linear polynomial.

23. Let $P(x)$ be a monic polynomial of degree n whose zeros are $i-1, i-2, \dots, i-n$ (where $i^2 = -1$) and let $R(x)$ and $S(x)$ be the real polynomials such that $P(x) = R(x) + iS(x)$. Prove that the polynomial $R(x)$ has n real zeros.

24. Let a, b, c be natural numbers. Prove that if there exist coprime polynomials P, Q, R with complex coefficients such that

$$P^a + Q^b = R^c,$$

then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$.

Corollary: The Last Fermat Theorem for polynomials.

25. Suppose that all zeros of a monic polynomial $P(x)$ with integer coefficients are of module 1. Prove that there are only finitely many such polynomials of any given degree; hence show that all its zeros are actually roots of unity, i.e. $P(x) \mid (x^n - 1)^k$ for some natural n, k.

9 Solutions

1. The polynomial $f(x)-10x$ vanishes at points $x=1,2,3$, so it is divisible by polynomial $(x-1)(x-2)(x-3)$. The monicity implies that $f(x)-10x=(x-1)(x-2)(x-3)(x-c)$ for some c. Now

$$f(12)+f(-8)=11\cdot 10\cdot 9\cdot(12-c)+120+(-9)(-10)(-11)(-8-c)-80=19840.$$

2. Note that $Q(x^2)=\prod(x^2-x_i^2)=\prod(x-x_i)\cdot\prod(x+x_i)=(-1)^nP(x)P(-x)$. We now have

$$b_1+b_2+\cdots+b_n=Q(1)-1=(-1)^nP(1)P(-1)-1=(-1)^n(1+B-A)(1+B+A),$$

where $A=a_1+a_3+a_5+\cdots$ and $B=a_2+a_4+\cdots$.

3. It follows from the conditions that $P(-\sin x)=P(\sin x)$, i.e. $P(-t)=P(t)$ for infinitely many t, so the polynomials $P(x)$ and $P(-x)$ coincide. Therefore, $P(x)=S(x^2)$ for some polynomial S. Now $S(\cos^2 x)=S(\sin^2 x)$ for all x, i.e. $S(1-t)=S(t)$ for infinitely many t, which implies $S(x)\equiv S(1-x)$. This is equivalent to $R(x-\frac{1}{2})=R(\frac{1}{2}-x)$, i.e. $R(y)\equiv R(-y)$, where R is a polynomial such that $S(x)=R(x-\frac{1}{2})$. Now $R(x)=T(x^2)$ for some polynomial T, and therefore $P(x)=S(x^2)=R(x^2-\frac{1}{2})=T(x^4-x^2+\frac{1}{4})=Q(x^4-x^2)$ for some polynomial Q.

4. (a) Clearly, $T_0(x)=1$ and $T_1(x)=x$ satisfy the requirements. For $n>1$ we use induction on n. Since $\cos(n+1)x=2\cos x\cos nx-\cos(n-1)x$, we can define $T_{n+1}=2T_1T_n-T_{n-1}$. Since T_1T_n and T_{n-1} are of degrees $n+1$ and $n-1$ respectively, T_{n+1} is of degree $n+1$ and has the leading coefficient $2\cdot 2^n=2^{n+1}$. It also follows from the construction that all its coefficients are integers.

   (b) The relation follows from the identity $\cos(m+n)x+\cos(m-n)x=2\cos mx\cos nx$.

   (c) The sequence of polynomials (U_n) satisfies $U_0(x)=2$, $U_1(x)=x$ and $U_{n+1}=U_1U_n-U_{n-1}$, implying that each U_n has integer coefficients. The equality $U_n(x+x^{-1})=x^n+x^{-n}$ holds for each $x=\cos t+i\sin t$, and therefore it holds for all x.

5. Suppose that $\cos\frac{p}{q}\pi=a$. It follows from the previous problem that $U_q(2a)=2\cos p\pi=\pm 2$, where U_q is monic with integer coefficients, so $2a$ is an integer by theorem 14.

6. Note that equality holds for a multiple of the n-th Chebyshev polynomial $T_n(x)$. The leading coefficient of T_n equals 2^{n-1}, so $C_n(x)=\frac{1}{2^{n-1}}T_n(x)$ is a monic polynomial and

$$|T_n(x)|=\frac{1}{2^{n-1}}|\cos(n\arccos x)|\le\frac{1}{2^{n-1}}\quad \text{za } x\in[-1,1].$$

Moreover, the values of T_n at points $1,\cos\frac{\pi}{n},\cos\frac{2\pi}{n},\cdots,\cos\frac{(n-1)\pi}{n},-1$ are alternately $\frac{1}{2^{n-1}}$ and $-\frac{1}{2^{n-1}}$.

Now suppose that $P\neq T_n$ is a monic polynomial such that $\max_{-1\le x\le 1}|P(x)|<\frac{1}{2^{n-1}}$. Then $P(x)-C_n(x)$ at points $1,\cos\frac{\pi}{n},\cdots,\cos\frac{(n-1)\pi}{n},-1$ alternately takes positive and negative values. Therefore the polynomial $P-C_n$ has at least n zeros, namely, at least one in every interval between two adjacent points. However, $P-C_n$ is a polynomial of degree $n-1$ as the monomial x^n is canceled, so we have arrived at a contradiction.

7. Since $P^{[i]}(x)=(-2)^{i-1}(-1)^x$ for $x=0,1,\dots,n-i$, we have

$$P(n+1)=P(n)+P^{[1]}(n-1)+\cdots+P^{[n]}(0)=\begin{cases}2^n, & 2\nmid n;\\ 1-2^n, & 2\mid n.\end{cases}$$

8. By theorem 20 we have
$$P(n+2)=\sum_{i=0}^{n}(-1)^{n-i}\frac{1}{i+1}\binom{n+1}{i}=\frac{1}{n+2}\sum_{i=0}^{n}(-1)^{n-i}\binom{n+2}{i+1}=\begin{cases}0, & 2\nmid n;\\ \frac{2}{n+2}, & 2\mid n.\end{cases}$$

9. By theorem 9, the polynomial $P(x)$ can be factorized as
$$P(x)=(x-a_1)^{\alpha_1}\cdots(x-a_k)^{\alpha_k}\cdot(x^2-b_1x+c_1)\cdots(x^2-b_mx+c_m), \qquad (*)$$
where a_i,b_j,c_j are real numbers such that the a_i are different and the polynomials $x^2-b_ix+c_i$ has no real zeros.

   The condition $P(x)\geq 0$ for all x implies that the α_i are even, whereas the condition $P(x)\geq 0$ for $x\geq 0$ implies that $(\forall i)$ α_i is even or $a_i<0$. It is now easy to write each factor in $(*)$ in the form A^2+B^2, respectively A^2+xB^2, so by the known formula $(a^2+\gamma b^2)(c^2+\gamma d^2)=(ac+\gamma bd)^2+\gamma(ad-bc)^2$ one can express their product $P(x)$ in the desired form.

10. Write
$$P(-x)=ax^4+(c-b)x^2+(e-d)-b(x^3-x^2)-d(x-1).$$
If r is a root of the polynomial Q, we have $P(\sqrt{r})=-(\sqrt{r}-1)(br+d)$ and $P(-\sqrt{r})=(\sqrt{r}+1)(br+d)$. Note that one of the two numbers $P(\pm\sqrt{r})$ positive and the other is negative (or both are zero). Hence there must be a zero of P between $-\sqrt{r}$ and $\sqrt{r}$.

11. Let us write $P(x)=(x-x_1)\cdots(x-x_m)(x^2-p_1x+q_1)\cdots(x^2-p_nx+q_n)$, where the polynomials $x^2-p_kx+q_k$ have no real zeros. We have
$$1>|P(i)|=\prod_{j=1}^{m}|i-x_j|\prod_{k=1}^{n}|-1-p_ki+q_k|,$$
and since $|i-x_j|^2=1+x_j^2>1$ for all j, we must have $|-1-p_ki+q_k|<1$ for some k, i.e.
$$p_k^2+(q_k-1)^2<1. \qquad (*)$$
Let $a\pm bi$ be the zeros of the polynomial $x^2-p_kx+q_k$ (and also of the polynomial P). Then $p_k=2a$ and $q_k=a^2+b^2$, so the inequality $(*)$ becomes $4a^2+(a^2+b^2-1)^2<1$, which is equivalent to the desired inequality.

12. Write f in the form $f=P/Q$, where P and Q are coprime polynomials and Q is monic. Comparing the leading coefficients we conclude that P is also monic. The condition of the problem becomes $P(x^2)/Q(x^2)=P(x)^2/Q(x)^2-a$. Since $P(x^2)$ and $Q(x^2)$ are coprime (if they have a common zero, so do P and Q), it follows that $Q(x^2)=Q(x)^2$ and hence $Q(x)=x^n$ for some $n\in\mathbb{N}$. Therefore, $P(x^2)=P(x)^2-ax^{2n}$.

    Let $P(x)=a_0+a_1x+\cdots+a_{m-1}x^{m-1}+x^m$. Comparing the coefficients of $P(x)^2$ and $P(x^2)$ we find that $a_{n-1}=\cdots=a_{2m-n+1}=0$, $a_{2m-n}=a/2$, $a_1=\cdots=a_{m-1}=0$ and $a_0=1$. Clearly, this is only possible if $a=0$, or $a=2$ and $2m-n=0$.

13. Since P is symmetric with respect to point 0, it is easy to show that P is also a polynomial in x^2, so there is a polynomial Q such that $P(x)=Q(x^2+1)$ or $P(x)=xQ(x^2+1)$. Then $Q((x^2+1)^2+1)=Q(x^2+1)^2-1$, respectively $(x^2+1)Q((x^2+1)^2+1)=x^2Q(x^2+1)^2+1$. The substitution $x^2+1=y$ yields $Q(y^2+1)=Q(y)^2+1$, resp. $yQ(y^2+1)=(y-1)Q(y)^2+1$.

    Suppose that $yQ(y^2+1)=(y-1)Q(y)^2+1$. Setting $y=1$ gives us $Q(2)=1$. Note that if $a\neq 0$ and $Q(a)=1$ then $aQ(a^2+1)=(a-1)+1$, so $Q(a^2+1)=1$ as well. This leads to an infinite sequence (a_n) of points at which Q takes the value 1, given by $a_0=2$ and $a_{n+1}=a_n^2+1$. We conclude that $Q\equiv 1$.

    We have shown that if $Q\not\equiv 1$, then $P(x)=Q(x^2+1)$. Now we easily come to all solutions: these are the polynomials of the form $T(T(\cdots(T(x))\cdots))$, where $T(x)=x^2+1$.

14. Let us denote $P(1) = a$. We have $a^2 - 2a - 2 = 0$. Since $P(x) = (x-1)P_1(x) + a$, substituting in the original equation and simplifying yields $(x-1)P_1(x)^2 + 2aP_1(x) = 4(x+1)P_1(2x^2 - 1)$. For $x = 1$ we have $2aP_1(1) = 8P_1(1)$, which together with $a \neq 4$ implies $P_1(1) = 0$, i.e. $P_1(x) = (x-1)P_2(x)$, so $P(x) = (x-1)^2P_2(x) + a$. Assume that $P(x) = (x-1)^nQ(x) + a$, where $Q(1) \neq 0$. Again substituting in the original equation and simplifying yields $(x-1)^nQ(x)^2 + 2aQ(x) = 2(2x+2)^nQ(2x^2-1)$, which implies that $Q(1) = 0$, a contradiction. We conclude that $P(x) = a$.

15. At first, note that there exists $x = a$ for which $P(a)^2 = Q(a)^2$. This follows from the fact that, if p and q are real roots of P and Q respectively, then $P(p)^2 - Q(p)^2 \leq 0 \leq P(q)^2 - Q(q)^2$, whereby $P^2 - Q^2$ is a continuous function. Then we also have $P(b) = Q(b)$ for $b = 1 + a + P(a)^2$. Assuming that a is the largest real number with $P(a) = Q(a)$, we come to an immediate contradiction.

16. Let $P(x) = a_0 + a_1x + \cdots + a_nx^n$. For every x the triple $(a,b,c) = (6x, 3x, -2x)$ satisfies the condition $ab + bc + ca = 0$. The condition in P gives us $P(3x) + P(5x) + P(-8x) = 2P(7x)$ for all x, so by comparing the coefficients on both sides we obtain $K(i) = \left(3^i + 5^i + (-8)^i - 2 \cdot 7^i\right) = 0$ whenever $a_i \neq 0$. Since $K(i)$ is negative for odd i and positive for $i = 0$ and even $i \geq 6$, $a_i = 0$ is only possible for $i = 2$ and $i = 4$. Therefore, $P(x) = a_2x^2 + a_4x^4$ for some real numbers a_2, a_4. It is easily verified that all such $P(x)$ satisfy the conditions.

17. Let d be the degree of P. There is a unique polynomial Q of degree at most d such that $Q(k) = a_k$ for $k = 1, 2, \ldots, d+1$. Let us show that $Q(n) = a_n$ for all n.

    Let $n > d+1$. Polynomial Q might not have integral coefficients, so we cannot deduce that $n - m \mid Q(n) - Q(m)$, but it certainly has rational coefficients, i.e. there is a natural number M for which $R(x) = MQ(x)$ has integral coefficients. By the condition of the problem, $M(a_n - Q(n)) = M(a_n - a_k) - (R(n) - R(k))$ is divisible by $n - k$ for each $k = 1, 2, \ldots, d+1$. Therefore, for each n we either have $a_n = Q(n)$ or

    $$L_n = \operatorname{lcm}(n-1, n-2, \ldots, n-d-1) \leq M(a_n - Q(n)) < Cn^d$$

    for some constant C independent of n.

    Suppose that $a_n \neq Q(n)$ for some n. note that L_n is not less than the product $(n-1)\cdots(n-d-1)$ divided by the product P of numbers $\gcd(n-i, n-j)$ over all pairs (i,j) of different numbers from $\{1, 2, \ldots, d+1\}$. Since $\gcd(n-i, n-j) \leq i - j$, we have $P \leq 1^d 2^{d-1} \cdots d$. It follows that

    $$(n-1)(n-2)\cdots(n-d-1) \leq PL_n < CPn^d,$$

    which is false for large enough n as the left hand side is of degree $d+1$. Thus, $a_n = Q(n)$ for each sufficiently large n, say $n > N$.

    What happens for $n \leq N$? By the condition of the problem, $M(a_n - Q(n)) = M(a_n - a_k) - (R(n) - R(k))$ is divisible by $m - n$ for every $m > N$, so it must be equal to zero. Hence $a_n = Q(n)$ for all n.

18. We have shown in 7 from the text that every such t satisfies $P(P(t)) = t$. If every such t also satisfies $P(t) = t$, the number of solutions is clearly at most $\deg P = n$. Suppose that $P(t_1) = t_2$, $P(t_2) = t_1$, $P(t_3) = t_4$ i $P(t_4) = t_3$, where $t_1 \neq t_{2,3,4}$. By theorem 10, $t_1 - t_3$ divides $t_2 - t_4$ and vice versa, from which we deduce that $t_1 - t_3 = \pm(t_2 - t_4)$. Assume that $t_1 - t_3 = t_2 - t_4$, i.e. $t_1 - t_2 = t_3 - t_4 = k \neq 0$. Since the relation $t_1 - t_4 = \pm(t_2 - t_3)$ similarly holds, we obtain $t_1 - t_3 + k = \pm(t_1 - t_3 - k)$ which is impossible. Therefore, we must have $t_1 - t_3 = t_4 - t_2$, which gives us $P(t_1) + t_1 = P(t_3) + t_3 = c$ for some c. It follows that all integral solutions t of the equation $P(P(t)) = t$ satisfy $P(t) + t = c$, and hence their number does not exceed n.

19. Suppose that $R = P - Q \neq 0$ and that $0 < k \leq n-1$ is the degree of $R(x)$. Then

$$P(P(x)) - Q(Q(x)) = [Q(P(x)) - Q(Q(x))] + R(P(x)).$$

Writing $Q(x) = x^n + \cdots + a_1x + a_0$ yields

$$Q(P(x)) - Q(Q(x)) = [P(x)^n - Q(x)^n] + \cdots + a_1[P(x) - Q(x)],$$

where all the summands but the first have a degree at most $n^2 - n$, while the first summand equals $R(x) \cdot \left(P(x)^{n-1} + P(x)^{n-2}Q(x) + \cdots + Q(x)^{n-1}\right)$ and has the degree $n^2 - n + k$ with the leading coefficient n. Therefore the degree of $Q(P(x)) - Q(Q(x))$ is $n^2 - n + k$. On the other hand, the degree of the polynomial $R(P(x))$ equals $kn < n^2 - n + k$, from which we conclude that the difference $P(P(x)) - Q(Q(x))$ has the degree $n^2 - n + k$, a contradiction.

It remains to check the case of a constant $R \equiv c$. Then the condition $P(P(x)) = Q(Q(x))$ yields $Q(Q(x) + c) = Q(Q(x)) - c$, so the equality $Q(y + c) = Q(y) - c$ holds for infinitely many values of y; hence $Q(y + c) \equiv Q(y) - c$ which is only possible for $c = 0$ (to see this, just compare the coefficients).

20. Suppose that $f(x) = g(x)h(x)$ for some nonconstant polynomials with integer coefficients. Since $|f(0)| = p$, either $|g(0)| = 1$ or $|h(0)| = 1$ holds. Assume w.l.o.g. that $|g(0)| = 1$. Write $g(x) = (x - \alpha_1) \cdots (x - \alpha_k)$. Then $|\alpha_1 \cdots \alpha_k| = 1$. Since $f(\alpha_i) - p = \alpha_i^m(\alpha_i - a)^n = -p$, taking the product over $i = 1, 2, \dots, k$ yields $|g(a)|^n = |(\alpha_1 - a) \cdots (\alpha_k - a)|^n = p^k$. Since $g(a)$ divides $|g(a)h(a)| = p$, we must have $|g(a)| = p$ and $n = k$. However, a must divide $|g(a) - g(0)| = p \pm 1$, which is impossible.

21. Suppose that $F = G \cdot H$ for some polynomials G, H with integer coefficients. Let us consider this equality modulo 2. Since $(x^2 + x + 1)^{2^n} \equiv F(x) \pmod 2$, we obtain $(x^2 + x + 1)^{2^n} = g(x)h(x)$, where $g \equiv G$ and $h \equiv H$ are polynomials over $\mathbb{Z}_2$. The polynomial $x^2 + x + 1$ is irreducible over $\mathbb{Z}_2[x]$, so there exists a natural number k for which $g(x) = (x^2 + x + 1)^k$ and $h(x) = (x^2 + x + 1)^{2^n - k}$; of course, these equalities hold in $\mathbb{Z}_2[x]$ only.

Back in $\mathbb{Z}[x]$, these equalities become $H(x) = (x^2 + x + 1)^{2^n - k} + 2V(x)$ and $G(x) = (x^2 + x + 1)^k + 2U(x)$ for some polynomials U and V with integer coefficients. Thus,

$$[(x^2 + x + 1)^k + 2U(x)][(x^2 + x + 1)^{2^n - k} + 2V(x)] = F(x).$$

Now if we set $x = \varepsilon = \frac{-1 + i\sqrt{3}}{2}$ in this equality, we obtain $U(\varepsilon)V(\varepsilon) = \frac{1}{4}F(\varepsilon) = \frac{1}{2}$. However, this is impossible as the polynomial $U(x)V(x)$ has integer coefficients, so $U(\varepsilon)V(\varepsilon)$ must be of the form $a + b\varepsilon$ for some $a, b \in \mathbb{Z}$ (since $\varepsilon^2 = -1 - \varepsilon$), which is not the case with $\frac{1}{2}$.

22. It is clear, for example by theorem 16, that P must have rational coefficients. For some $m \in \mathbb{N}$ the coefficients of the polynomial $mP(x)$ are integral. Let p be a prime number not dividing m. We claim that, if P is not linear, there is no rational number x for which $P(x) = \frac{1}{mp}$. Namely, such an x would also satisfy $Q(x) = mpP(x) - 1 = 0$. On the other hand, the polynomial $Q(x)$ is irreducible because so is the polynomial $x^nQ(1/x)$ by the Eisenstain criterion; indeed, all the coefficients of $x^nQ(1/x)$ but the first are divisible by p and the constant term is not divisible by p^2. This proves our claim.

23. Denote $P(x) = P_n(x) = R_n(x) + iS_n(x)$. We prove by induction on n that all zeros of P_n are real; moreover, if $x_1 > x_2 > \cdots > x_n$ are the zeros of R_n and $y_1 > y_2 > \cdots > y_{n-1}$ the zeros of R_{n-1}, then

$$x_1 > y_1 > x_2 > y_2 > \cdots > x_{n-1} > y_{n-1} > x_n.$$

This statement is trivially true for $n = 1$. Suppose that it is true for $n - 1$.

Since $R_n + iS_n = (x-i+n)(R_{n-1}+iS_{n-1})$, the polynomials R_n and S_n satisfy the recurrent relations $R_n = (x+n)R_{n-1}+S_{n-1}$ and $S_n = (x+n)S_{n-1}-R_{n-1}$. This gives us

$$R_n - (2x+2n-1)R_{n-1} + [(x+n-1)^2+1]R_{n-2} = 0.$$

If $z_1 > \cdots > z_{n-2}$ are the (real) zeros R_{n-2}, by the inductive hypothesis we have $z_{i-1} > y_i > z_i$. Since the value of R_{n-2} is alternately positive and negative on the intervals $(z_1,+\infty)$, (z_2,z_1), etc, it follows that $\operatorname{sgn} R_{n-2}(y_i) = (-1)^{i-1}$. Now we conclude from the relation $R_n(y_i) = -[(x+n-1)^2+1]R_{n-2}(y_i)$ that

$$\operatorname{sgn} R_n(y_i) = (-1)^i,$$

which means that the polynomial R_n has a zero on each of the n intervals $(y_1,+\infty)$, (y_2,y_1), $\dots$, $(-\infty,y_{n-1})$. This finishes the induction.

24. We first prove the following auxiliary statement.

Lemma. If A,B and C are coprime polynomials with $A+B=C$, then the degree of each of the polynomials A,B,C is less than the number of different zeros of the polynomial ABC.

Proof. Let

$$A(x) = \prod_{i=1}^{k}(x-p_i)^{a_i}, \quad B(x) = \prod_{i=1}^{l}(x-q_i)^{b_i}, \quad C(x) = \prod_{i=1}^{m}(x-r_i)^{c_i}.$$

Let us rewrite the given equality as $A(x)/C(x) + B(x)/C(x) = 1$ and differentiate it with respect to x. We obtain

$$\frac{A(x)}{C(x)}\left(\sum_{i=1}^{k}\frac{a_i}{x-p_i} - \sum_{i=1}^{m}\frac{c_i}{x-r_i}\right) = -\frac{B(x)}{C(x)}\left(\sum_{i=1}^{l}\frac{b_i}{x-q_i} - \sum_{i=1}^{m}\frac{c_i}{x-r_i}\right),$$

from which we see that $A(x)/B(x)$ can be expressed as a quotient of two polynomials of degree not exceeding $k+l+m-1$. The statement follows from the coprimeness of A and B.

Now we apply the Lemma on the polynomials P^a, Q^b, R^c. We obtain that each of the numbers $a\deg P$, $b\deg Q$, $c\deg R$ is less than $\deg P + \deg Q + \deg R$, and therefore

$$\frac{1}{a} > \frac{\deg P}{\deg P + \deg Q + \deg R},$$

etc. Adding these yields the desired inequality.

25. Let us fix $\deg P = n$. Let $P(x) = (x-z_1)\cdots(x-z_n) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $|z_i| = 1$ for $i = 1,\dots,n$. By the Vieta formulas, $a_{n-i} = \pm\sigma_i(z_1,\dots,z_n)$, which is a sum of $\binom{n}{i}$ summands of modulus 1, and hence $|a_{n-i}| \le \binom{n}{i}$. Therefore, there are at most $2\binom{m}{i}+1$ possible values of the coefficient of $P(x)$ at x^{n-i} for each i. Thus the number of possible polynomials P of degree n is finite.

Now consider the polynomial $P_r(x) = (x-z_1^r)\cdots(x-z_n^r)$ for each natural number r. All coefficients of polynomial P_r are symmetric polynomials in z_i with integral coefficients, so by the theorem 24 they must be integers. Therefore, every polynomial P_r satisfies the conditions of the problem, but there are infinitely many r's and only finitely many such polynomials. We conclude that $P_r(x) = P_s(x)$ for some distinct $r,s \in \mathbb{N}$, and the main statement of the problem follows.

Generating Functions

Milan Novaković

Contents

1 Introduction

Generating functions are powerful tools for solving a number of problems mostly in combinatorics, but can be useful in other branches of mathematics as well. The goal of this text is to present certain applications of the method, and mostly those using the high school knowledge.

In the beginning we have a formal treatement of generating functions, i.e. power series. In other parts of the article the style of writing is more problem-soving oriented. First we will focus on solving the reccurent equations of first, second, and higher order, after that develope the powerful method of ,,the snake oil“, and for the end we leave some other applications and various problems where generating functions can be used.

The set of natural numbers will be denoted by $\mathbb{N}$, while $\mathbb{N}_0$ will stand for the set of non-negative integers. For the sums going from 0 to $+\infty$ the bounds will frequently be omitted – if a sum is without the bounds, they are assumed to be 0 and $+\infty$.

2 Theoretical Introduction

In dealing with generating functions we frequently want to use different transformations and manipulations that are illegal if the generating functions are viewed as analytic functions. Therefore they will be introduced as algebraic objects in order to obtain wider range of available methods. The theory we will develope is called the *formal theory of power series*.

Definition 1. A formal power series *is the expression of the form*

$$a_0 + a_1x + a_2x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i.$$

A sequence of integers $\{a_i\}_0^\infty$ *is called* the sequence of coefficients.

Remark. We will use the other expressions also: series, generating function...

For example the series

$$A(x) = 1 + x + 2^2x^2 + 3^3x^3 + \cdots + n^nx^n + \cdots$$

converges only for $x=0$ while, in the formal theory this is well defined formal power series with the correspongind sequence of coefficients equal to $\{a_i\}_0^\infty, a_i = i^i$.
Remark. Sequences and their elements will be most often denoted by lower-case latin letters (a, b, $a_3 \cdots$), while the power series generated by them (unless stated otherwise) will be denoted by the corresponding capital letters (A, B, $\cdots$).

Definition 2. *Two series* $A=\sum_{i=0}^{\infty} a_i x^i$ *and* $B=\sum_{i=0}^{\infty} b_i x^i$ *are called* equal *if their corresponding sequences of coefficients are equal, i.e.* $a_i = b_i$ *for every* $i \in \mathbb{N}_0$.

Remark. The coefficient near x^n in the power series F will be denoted by $[x^n]F$.

We can define the *sum* and the *difference* of power series in the following way

$$\sum_n a_n x^n \pm \sum_n b_n x^n = \sum_n (a_n \pm b_n) x^n$$

while the *product* is defined by

$$\sum_n a_n x^n \sum_n b_n x^n = \sum_n c_n x^n, \qquad c_n = \sum_i a_i b_{n-i}$$

Instead of $F \cdot F$ we write F^2, and more generally $F^{n+1} = F \cdot F^n$. We see that the neutral for addition is 0, and 1 is the neutral for multiplication. Now we can define the following term:

Definition 3. *The formal power series G is* reciprocal *to the formal power series F if $FG=1$.*

The generating function reciprocal to F will be usually denoted by $1/F$. Since the multiplication is commutative we have that $FG=1$ is equivalent to $GF=1$ hence F and G are *mutually reciprocal*. We also have $(1-x)(1+x+x^2+\cdots) = 1+\sum_{i=1}^{\infty}(1\cdot 1 - 1\cdot 1)x^i = 1$ hence $(1-x)$ and $(1+x+x^2+\cdots)$ are mutually reciprocal.

Theorem 1. *Formal power series* $F=\sum_n a_n x^n$ *has a reciprocal if and only if* $a_0 \neq 0$. *In that case the reciprocal is unique.*

Proof. Assume that F has a reciprocal given by $1/F = \sum_n b_n x^n$. Then $F\cdot(1/F)=1$ implying $1=a_0b_0$ hence $a_0 \neq 0$. For $n \geqslant 1$ we have $0=\sum_k a_k b_{n-k}$ from where we conclude.

$$b_n = -\frac{1}{a_0}\sum_k a_k b_{n-k}.$$

The coefficients are uniquely determined by the prefious formula.

On the other hand if $a_0 \neq 0$ we can uniquely determine all coefficients b_i using the previously established relations which gives the series $1/F$. □

Now we can conclude that the set of power series with the above defined operation forms a ring whose invertible elements are precisely those power series with the non-zero first coefficient.

If $F=\sum_n f_n x^n$ is a power series, $F(G(x))$ will denote the series $F(G(x)) = \sum_n f_n G(x)^n$. This notation will be used also in the case when F is a polynomial (i.e. when there are only finitely many non-zero coefficients) or if the free term of G equals 0. In the case that the free term of G equal to 0, and F is not a polynomial, we can't determine the particular element of the series $F(G(x))$ in finitely many steps.

Definition 4. *A formal power series G is said to be an inverse of F if $F(G(x)) = G(F(x)) = x$.*

We have a symmetry here as well, if G is inverse of F than F is inverse of G as well.

Theorem 2. *Let F and G be mutually inverse power series. Then $F = f_1x + f_2x^2 + \cdots$, $G = g_1x + g_2x^2 + \cdots$, and $f_1g_1 \neq 0$.*

Proof. In order for $F(G(x))$ and $G(F(x))$ to be defined we must have 0 free terms. Assume that $F = f_rx^r + \cdots$ and $G = g_sx^s + \cdots$. Then $F(G(x)) = x = f_rg_s^rx^{rs} + \cdots$, thus $rs = 1$ and $r = s = 1$. □

Definition 5. *The derivative of a power series $F = \sum_n f_nx^n$ is $F' = \sum_n nf_nx^{n-1}$. The derivative of order $n > 1$ is defined recursively by $F^{(n+1)} = (F^{(n)})'$.*

Theorem 3. *The following properties of the derivative hold:*

- $(F+G)^{(n)} = F^{(n)} + G^{(n)}$
- $(FG)^{(n)} = \sum_{i=0}^{n}\binom{n}{i}F^{(i)}G^{(n-i)}$

The proof is very standard as is left to the reader. □

We will frequently associate the power series with its generating sequence, and to make writing more clear we will define the the relation $\overset{osr}{\leftrightarrow}$ in the following way:

Definition 6. *$A \overset{osr}{\leftrightarrow} \{a_n\}_0^\infty$ means that A is a usual power series which is generated by $\{a_n\}_0^\infty$, i.e. $A = \sum_n a_nx^n$.*

Assume that $A \overset{osr}{\leftrightarrow} \{a_n\}_0^\infty$. Then

$$\sum_n a_{n+1}x^n = \frac{1}{x}\sum_{n>0} a_nx^n = \frac{A(x) - a_0}{x}$$

or equivalently $\{a_{n+1}\}_0^\infty \overset{osr}{\leftrightarrow} \frac{A - a_0}{x}$. Similarly

$$\{a_{n+2}\}_0^\infty \overset{osr}{\leftrightarrow} \frac{(A-a_0)/x - a_1}{x} = \frac{A - a_0 - a_1x}{x^2}.$$

Theorem 4. *If $\{a_n\}_0^\infty \overset{osr}{\leftrightarrow} A$ the for $h > 0$:*

$$\{a_{n+h}\}_0^\infty \overset{osr}{\leftrightarrow} \frac{A - a_0 - a_1x - \cdots - a_{h-1}x^{h-1}}{x^h}.$$

Proof. We will use the induction on h. For $h = 1$ the statement is true and that is shown before. If the statement holds for some h then

$$\begin{aligned}\{a_{n+h+1}\}_0^\infty &\overset{osr}{\leftrightarrow} \{a_{(n+h)+1}\}_0^\infty \overset{osr}{\leftrightarrow} \frac{\frac{A - a_0 - a_1x - \cdots - a_{h-1}x^{h-1}}{x^h} - a_h}{x} \\ &\overset{osr}{\leftrightarrow} \frac{A - a_0 - a_1x - \cdots - a_hx^h}{x^{h+1}},\end{aligned}$$

which finishes the proof. □

We already know that $\{(n+1)a_{n+1}\}_0^\infty \overset{osr}{\leftrightarrow} A'$. Our goal is to obtain the sequence $\{na_n\}_0^\infty$. That is exactly the sequence xA'. We will define the operator xD in the following way:

Definition 7. *$xDA = xA'$ i.e. $xDA = x\frac{dA}{dx}$.*

The following two theorems are obvious consequences of the properties of the derivative:

Theorem 5. *Let $\{a_n\}_0^\infty \overset{osr}{\leftrightarrow} A$. Then $\{n^ka_n\}_0^\infty \overset{osr}{\leftrightarrow} (xD)^kA$.*

Theorem 6. *Let $\{a_n\}_0^\infty \overset{osr}{\leftrightarrow} A$ and P be a polynomial. Then*

$$P(xD)A \overset{osr}{\leftrightarrow} \{P(n)a_n\}_0^\infty$$

Let us consider the generating function $\dfrac{A}{1-x}$. It can be written as $A\frac{1}{1-x}$. As we have shown before the reciprocal to the series $1-x$ is $1+x+x^2+\cdots$, hence $\frac{A}{1-x} = (a_0+a_1x+a_2x^2+\cdots)(1+x+x^2+\cdots) = a_0+(a_0+a_1)x+(a_0+a_1+a_2)x^2+\cdots$.

Theorem 7. *If $\{a_n\}_0^\infty \overset{osr}{\leftrightarrow} A$ then*

$$\frac{A}{1-x} \quad \overset{osr}{\leftrightarrow} \quad \left\{\sum_{j=0}^{n} a_j\right\}_{n\geqslant 0}.$$

Now we will introduce the new form of generating functions.

Definition 8. *We say that A is* exponential generating function *(or series, power series) of the sequence $\{a_n\}_0^\infty$ if A is the ususal generating function of the sequence $\{\frac{a_n}{n!}\}_0^\infty$, or equivalently*

$$A = \sum_n \frac{a_n}{n!}x^n.$$

If B is exponential generating function of the series $\{b_n\}_0^\infty$ we can also write $\{b_n\}_0^\infty \overset{esr}{\leftrightarrow} B$.

If $B \overset{esr}{\leftrightarrow} \{b_n\}_0^\infty$, we are interested in B'. It is easy to see that

$$B' = \sum_{n=1}^{\infty} \frac{nb_nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{b_nx^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{b_{n+1}x^n}{n!}$$

hence $B' \overset{esr}{\leftrightarrow} \{b_{n+1}\}_0^\infty$.

Theorem 8. *If $\{b_n\}_0^\infty \overset{esr}{\leftrightarrow} B$ then for $h \geqslant 0$:*

$$\{b_{n+h}\}_0^\infty \quad \overset{osr}{\leftrightarrow} \quad B^{(h)}.$$

We also have an equivalent theorem for exponential generating functions.

Theorem 9. *Let $\{b_n\}_0^\infty \overset{esr}{\leftrightarrow} B$ and let P be a polynomial. Then*

$$P(xD)B \overset{esr}{\leftrightarrow} \{P(n)b_n\}_0^\infty$$

The exponential generating functions are useful in combinatorial identities because of the following property.

Theorem 10. *Let $\{a_n\}_0^\infty \overset{esr}{\leftrightarrow} A$ and $\{b_n\}_0^\infty \overset{esr}{\leftrightarrow} B$. Then the generating function AB generates the sequence*

$$\left\{\sum_k \binom{n}{k} a_k b_{n-k}\right\}_{n=0}^{\infty}.$$

Proof. We have that

$$AB = \left\{\sum_{i=0}^{\infty} \frac{a_ix^i}{i!}\right\}\left\{\sum_{j=0}^{\infty} \frac{b_jx^j}{j!}\right\} = \sum_{i,j\geqslant 0} \frac{a_ib_j}{i!j!}x^{i+j} = \sum_n x^n \left\{\sum_{i+j=n} \frac{a_ib_j}{i!j!}\right\},$$

or

$$AB = \sum_n \frac{x^n}{n!}\left\{\sum_{i+j=n} \frac{n!a_ib_j}{i!j!}\right\} = \sum_n \frac{x^n}{n!}\sum_k \binom{n}{k} a_kb_{n-k},$$

and the proof is complete. □

We have listed above the fundamental properties of generating functions. New properties and terms will be defined later.

Although the formal power series are defined as solely algebraic objects, we aren't giving up their analytical properties. We will use the well-known Taylor's expansions of functions into power series. For example, we will treat the function e^x as a formal power series obtained by expanding the function into power series, i.e. we will identify e^x with $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. We will use the converse direction also. Here we will list the Taylor expansions of most common functions.

$$\frac{1}{1-x} = \sum_{n \geqslant 0} x^n$$

$$\ln \frac{1}{1-x} = \sum_{n \geqslant 1} \frac{x^n}{n}$$

$$e^x = \sum_{n \geqslant 0} \frac{x^n}{n!}$$

$$\sin x = \sum_{n \geqslant 0} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n \geqslant 0} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$(1+x)^\alpha = \sum_k \binom{\alpha}{k} x^k$$

$$\frac{1}{(1-x)^{k+1}} = \sum_n \binom{n+k}{n} x^n$$

$$\frac{x}{e^x - 1} = \sum_{n \geqslant 0} \frac{B_n x^n}{n!}$$

$$\arctan x = \sum_{n \geqslant 0} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\frac{1}{2x}(1 - \sqrt{1-4x}) = \sum_n \frac{1}{n+1} \binom{2n}{n} x^n$$

$$\frac{1}{\sqrt{1-4x}} = \sum_n \binom{2n}{n} x^n$$

$$x \cot x = \sum_{n \geqslant 0} \frac{(-4)^n B_{2n}}{(2k)!} x^{2n}$$

$$\tan x = \sum_{n \geqslant 1} (-1)^{n-1} \frac{2^{2n}(2^{2n}-1) B_{2n}}{(2n)!} x^{2n-1}$$

$$\frac{x}{\sin x} = \sum_{n \geqslant 0} (-1)^{n-1} \frac{(4^n - 2) B_{2n}}{(2n)!} x^{2n}$$

$$\frac{1}{\sqrt{1-4x}} \left(\frac{1 - \sqrt{1-4x}}{2x} \right)^k = \sum_n \binom{2n+k}{n} x^n$$

$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^k = \sum_{n\geqslant 0}\frac{k(2n+k-1)!}{n!(n+k)!}x^n$$

$$\arcsin x = \sum_{n\geqslant 0}\frac{(2n-1)!!x^{2n+1}}{(2n)!!(2n+1)}$$

$$e^x \sin x = \sum_{n\geqslant 1}\frac{2^{\frac{n}{2}}\sin\frac{n\pi}{4}}{n!}x^n$$

$$\ln^2\frac{1}{1-x} = \sum_{n\geqslant 2}\frac{H_{n-1}}{n}x^n$$

$$\sqrt{\frac{1-\sqrt{1-x}}{x}} = \sum_{n=0}^{\infty}\frac{(4n)!}{16^n\sqrt{2}(2n)!(2n+1)!}x^n$$

$$\left(\frac{\arcsin x}{x}\right)^2 = \sum_{n=0}^{\infty}\frac{4^n n!^2}{(k+1)(2k+1)!}x^{2n}$$

Remark: Here $H_n = \sum_{i=1}^{n}\frac{1}{i}$, and B_n is the n-th Bernoulli number.

3 Recurrent Equations

We will first solve one basic recurrent equation.

Problem 1. *Let a_n be a sequence given by $a_0 = 0$ and $a_{n+1} = 2a_n + 1$ for $n \geqslant 0$. Find the general term of the sequence a_n.*

Solution. We can calculate the first several terms 0, 1, 3, 7, 15, and we are tempted to guess the solution as $a_n = 2^n - 1$. The previous formula can be easily established using mathematical induction but we will solve the problem using generating functions. Let $A(x)$ be the generating function of the sequence a_n, i.e. let $A(x) = \sum_n a_n x^n$. If we multiply both sides of the recurrent relation by x^n and add for all n we get

$$\sum_n a_{n+1}x^n = \frac{A(x)-a_0}{x} = \frac{A(x)}{x} = 2A(x) + \frac{1}{1-x} = \sum_n (2a_n+1)x^n.$$

From there we easily conclude

$$A(x) = \frac{x}{(1-x)(2-x)}.$$

Now the problem is obtaining the general formula for the elements of the sequence. Here we will use the famous trick of decomposing A into two fractions each of which will have the corresponding generating function. More precisely

$$\frac{x}{(1-x)(2-x)} = x\left(\frac{2}{1-2x} - \frac{1}{1-x}\right) = (2x + 2^2x^2 + \cdots) - (x + x^2 + \cdots).$$

Now it is obvious that $A(x) = \sum_{n=0}^{\infty}(2^n-1)x^n$ and the solution to the recurrent relation is indeed $a_n = 2^n - 1$. △

Problem 2. *Find the general term of the sequence given recurrently by*

$$a_{n+1} = 2a_n + n, \quad (n \geqslant 0), \;\; a_0 = 1.$$

Solution. Let $\{a_n\}_0^\infty \overset{osr}{\leftrightarrow} A$. Then $\{a_{n+1}\}_0^\infty \overset{osr}{\leftrightarrow} \frac{A-1}{x}$. We also have that $xD\frac{1}{1-x} \overset{osr}{\leftrightarrow} \{n \cdot 1\}$. Since $xD\frac{1}{1-x} = x\frac{1}{(1-x)^2} = \frac{x}{(1-x)^2}$ the recurrent relation becomes

$$\frac{A-1}{x} = 2A + \frac{x}{(1-x)^2}.$$

From here we deduce

$$A = \frac{1-2x+2x^2}{(1-x)^2(1-2x)}.$$

Now we consider that we have *solved for the generating series*. If we wanted to show that the sequence is equal to some other sequence it would be enough to show that the functions are equal. However we need to find the terms explicitely. Let us try to represent A again in the form

$$\frac{1-2x+2x^2}{(1-x)^2(1-2x)} = \frac{P}{(1-x)^2} + \frac{Q}{1-x} + \frac{R}{1-2x}.$$

After multiplying both sides with $(1-x)^2(1-2x)$ we get

$$1-2x+2x^2 = P(1-2x) + Q(1-x)(1-2x) + R(1-x)^2,$$

or equivalently

$$1-2x+2x^2 = x^2(2Q+R) + x(-2P-3Q-2R) + (P+Q+R).$$

This implies $P=-1$, $Q=0$, and $R=2$. There was an easier way to get P, Q, and R. If we multiply both sides by $(1-x)^2$ and set $x=1$ we get $P=-1$. Similarly if we multiply everything by $1-2x$ and plug $x=\frac{1}{2}$ we get $R=2$. Now substituting P and R and setting $x=0$ we get $Q=0$.

Thus we have

$$A = \frac{-1}{(1-x)^2} + \frac{2}{1-2x}.$$

Since $\frac{2}{1-2x} \overset{osr}{\leftrightarrow} \{2^{n+1}\}$ and $\frac{1}{(1-x)^2} = D\frac{1}{1-x} \overset{osr}{\leftrightarrow} \{n+1\}$ we get $a_n = 2^{n+1}-n-1$. $\triangle$

In previous two examples the term of the sequence was depending only on the previous term. We can use generating functions to solve recurrent relations of order greater than 1.

Problem 3 (Fibonacci's sequence). $F_0=0$, $F_1=1$, *and for* $n \geqslant 1$, $F_{n+1} = F_n + F_{n-1}$. *Find the general term of the sequence.*

Solution. Let F be the generating function of the series $\{F_n\}$. If we multiply both sides by x^n and add them all, the left-hand side becomes $\{F_{n+1}\} \overset{osr}{\leftrightarrow} \frac{F-x}{x}$, while the right-hand side becomes $F+xF$. Therefore

$$F = \frac{x}{1-x-x^2}.$$

Now we want to expand this function into power series. First we want to represent the function as a sum of two fractions. Let

$$-x^2-x+1 = (1-\alpha x)(1-\beta x).$$

Then $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, and $\alpha-\beta = \sqrt{5}$. We further have

$$\begin{aligned}\frac{x}{1-x-x^2} &= \frac{x}{(1-x\alpha)(1-x\beta)} = \frac{1}{\alpha-\beta}\left(\frac{1}{1-x\alpha} - \frac{1}{1-x\beta}\right)\\ &= \frac{1}{\sqrt{5}}\left\{\sum_{n=0}^{\infty}\alpha^n x^n - \sum_{n=0}^{\infty}\beta^n x^n\right\}.\end{aligned}$$

It is easy to obtain

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n). \triangle$$

Remark: From here we can immediately get the approximate formula for F_n. Since $|\beta| < 1$ we have $\lim_{n\to\infty} \beta^n = 0$ and

$$F_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

Now we will consider the case with the sequence of two variables.

Problem 4. *Find the number of k-element subsets of an n-element set.*

Solution. We know that the result is $\binom{n}{k}$, but we want to obtain this using the generating functions. Assume that the required number is equal to $c(n,k)$. Let $A = \{a_1,\dots,a_n\}$ be an n-element set. There are two types of k-element subsets – those which contain a_n and those that don't. There are exactly $c(n-1,k-1)$ subsets containing a_n. Indeed they are all formed by taking $k-1$-element subsets of $\{a_1,\dots,a_{n-1}\}$ and adding a_n to each of them. On the other hand there are exaclty $c(n-1,k)$ subsets not contianing a_n. Hence

$$c(n,k) = c(n-1,k) + c(n-1,k-1).$$

We also have $c(n,0) = 1$. Now we will define the generating function of the sequence $c(n,k)$ for a fixed n. Assume that

$$C_n(x) = \sum_k c(n,k)x^k.$$

If we multiply the recurrent relation by x^k and add for all $k \geqslant 1$ we get

$$C_n(x) - 1 = (C_{n-1}(x) - 1) + xC_{n-1}(x), \text{ for each } n \geqslant 0$$

and $C_0(x) = 1$. Now we have for $n \geqslant 1$:

$$C_n(x) = (1+x)C_{n-1}(x).$$

We finally have $C_n(x) = (1+x)^n$. Hence, $c(n,k)$ is the coefficient near x^k in the expansion of $(1+x)^n$, and that is exactly $\binom{n}{k}$. $\triangle$

Someone might think that this was a cheating. We have used binomial formula, and that is obtianed using a combinatorial technique which uses the result we wanted to prove. Fortunately, there is a proof of binomial formula involving Taylor expainsion.

We can also make a generating function of the sequece $C_n(x)$:

$$\sum_n C_n(x)y^n = \sum_n\sum_k \binom{n}{k}x^ky^n = \sum_n (1+x)^ny^n = \frac{1}{1-y(1+x)}.$$

In such a way we have $\binom{n}{k} = [x^ky^n](1-y(1+x))^{-1}$. Now we can calculate the sum $\sum_n \binom{n}{k}y^n$:

$$\begin{aligned}[x^k]\sum_n\sum_k \binom{n}{k}x^ky^n &= [x^k]\frac{1}{1-y(1+x)} = \frac{1}{1-y}[x^k]\frac{1}{1-\frac{y}{1-y}x} \\ &= \frac{1}{1-y}\left(\frac{y}{1-y}\right)^k = \frac{y^k}{(1-y)^{k+1}}.\end{aligned}$$

Hence we have the identities

$$\sum_k \binom{n}{k}x^k = (1+x)^n; \qquad \sum_n \binom{n}{k}y^n = \frac{y^k}{(1-y)^{k+1}}.$$

Remark: For $n < k$ we define $\binom{n}{k} = 0$.

Problem 5. *Find the general term of the sequence* $a_{n+3} = 6a_{n+2} - 11a_{n+1} + 6a_n$, $n \geqslant 0$ *with the initial conditions* $a_0 = 2$, $a_1 = 0$, $a_2 = -2$.

Solution. If A is the generating function of the corresponding sequence then:

$$\frac{A - 2 - 0\cdot x - (-2)x^2}{x^3} = 6\frac{A - 2 - 0\cdot x}{x^2} - 11\frac{A-2}{x} + 6A,$$

from where we easily get

$$A = \frac{20x^2 - 12x + 2}{1 - 6x + 11x^2 - 6x^3} = \frac{20x^2 - 12x + 2}{(1-x)(1-2x)(1-3x)}.$$

We want to find the real coefficients B, C, and D such that

$$\frac{20x^2 - 12x + 2}{(1-x)(1-2x)(1-3x)} = \frac{B}{1-x} + \frac{C}{1-2x} + \frac{D}{1-3x}.$$

We will multiply both sides by $(1-x)$ and set $x = 1$ to obtain $B = \frac{20-12+2}{(-1)\cdot(-2)} = 5$. Multiplying by $(1-2x)$ and setting $x = 1/2$ we further get $C = \frac{5-6+2}{-\frac{1}{4}} = -4$. If we now substitute the found values for B and C and put $x = 0$ we get $B + C + D = 2$ from where we deduce $D = 1$. We finally have

$$A = \frac{5}{1-x} - \frac{4}{1-2x} + \frac{1}{1-3x} = \sum_{n=0}^{\infty}(5 - 4\cdot 2^n + 3^n)x^n$$

implying $a_n = 5 - 2^{n+2} + 3^n$. $\triangle$

The following example will show that sometimes we can have troubles in finding the explicite formula for the elements of the sequence.

Problem 6. *Let the sequence be given by* $a_0 = 0$, $a_1 = 2$, *and for* $n \leqslant 0$:

$$a_{n+2} = -4a_{n+1} - 8a_n.$$

Find the general term of the sequence.

Solution. Let A be the generating function of the sequence. The recurrent relation can be written in the form

$$\frac{A - 0 - 2x}{x^2} = -4\frac{A-0}{x} - 8A$$

implying

$$A = \frac{2x}{1 + 4x + 8x^2}.$$

The roots $r_1 = -2 + 2i$ and $r_2 = -2 - 2i$ of the equation $x^2 + 4x + 8$ are not real. However this should interfere too much with our intention for finding B and C. Pretending that nothing wierd is going on we get

$$\frac{2x}{1 + 4x + 8x^2} = \frac{B}{1 - r_1 x} + \frac{C}{1 - r_2 x}.$$

Using the trick learned above we get $B = \frac{-i}{2}$ and $C = \frac{i}{2}$.

Did you read everything carefully? Why did we consider the roots of the polynomial $x^2 + 4x + 8$ when the denumerator of A is $8x^2 + 4x + 1$?! Well if we had considered the roots of the real denumerator we would get the fractions of the form $\frac{B}{r_1 - x}$ which could give us a trouble if we wanted to use power series. However we can express the denominator as $x^2(8 + 4\frac{1}{x} + \frac{1}{x^2})$ and consider this as a polynomial in $\frac{1}{x}$! Then the denumerator becomes $x^2\left(\frac{1}{x} - r_1\right)\cdot\left(\frac{1}{x} - r_2\right)$.

Now we can proceed with solving the problem. We get

$$A=\frac{-i/2}{1-(-2+2i)x}+\frac{i/2}{1-(-2-2i)x}.$$

From here we get

$$A=\frac{-i}{2}\sum_{n=0}^{\infty}(-2+2i)^n x^n+\frac{i}{2}\sum_{n=0}^{\infty}(-2-2i)^n x^n,$$

implying

$$a_n=\frac{-i}{2}(-2+2i)^n+\frac{i}{2}(-2-2i)^n.$$

But the terms of the sequence are real, not complex numbers! We can now simplify the expression for a_n. Since

$$-2\pm 2i=2\sqrt{2}e^{\frac{\pm 3\pi i}{4}},$$

we get

$$a_n=\frac{i}{2}(2\sqrt{2})^n\left((\cos\frac{3n\pi}{4}-i\sin\frac{3n\pi}{4})-(\cos\frac{3n\pi}{4}+i\sin\frac{3n\pi}{4})\right),$$

hence $a_n=(2\sqrt{2})^n\sin\frac{3n\pi}{4}$. Written in another way we get

$$a_n=\begin{cases} 0, & n=8k\\ (2\sqrt{2})^n, & n=8k+6\\ -(2\sqrt{2})^n, & n=8k+2\\ \frac{1}{\sqrt{2}}(2\sqrt{2})^n, & n=8k+1 \text{ ili } n=8k+3\\ -\frac{1}{\sqrt{2}}(2\sqrt{2})^n, & n=8k+5 \text{ ili } n=8k+7.\ \triangle \end{cases}$$

Now we will consider on more complex recurrent equation.

Problem 7. *Find the general term of the sequence x_n given by:*

$$x_0=x_1=0,\qquad x_{n+2}-6x_{n+1}+9x_n=2^n+n\quad za\ n\geqslant 0.$$

Solution. Let $X(t)$ be the generating function of our sequence. Using the same methods as in the examples above we can see that the following holds:

$$\frac{X}{t^2}-6\frac{X}{t}+9X=\frac{1}{1-2t}+\frac{t}{(1-t)^2}.$$

Simplifying the expression we get

$$X(t)=\frac{t^2-t^3-t^4}{(1-t)^2(1-2t)(1-3t)^2},$$

hence

$$X(t)=\frac{1}{4(1-x)^2}+\frac{1}{1-2x}-\frac{5}{3(1-3x)}+\frac{5}{12(1-3x)^2}.$$

The sequence corresponding to the first summand is $\frac{n+1}{4}$, while the sequences for the second, third, and fourth are 2^n, $5\cdot 3^{n-1}$, and $\frac{5(n+1)3^{n+1}}{12}$ respectively. Now we have

$$x_n=\frac{2^{n+2}+n+1+5(n-3)3^{n-1}}{4}.\ \triangle$$

Problem 8. *Let $f_1 = 1$, $f_{2n} = f_n$, and $f_{2n+1} = f_n + f_{n+1}$. Find the general term of the sequence.*

Solution. We see that the sequence is well define because each term is defined using the terms already defined. Let the generating function F be given by

$$F(x) = \sum_{n\geqslant 1} f_n x^{n-1}.$$

Multiplying the first given relation by x^{2n-1}, the second by x^{2n}, and adding all of them for $n \geq 1$ we get:

$$f_1 + \sum_{n\geqslant 1} f_{2n}x^{2n-1} + \sum_{n\geqslant 1} f_{2n+1}x^{2n} = 1 + \sum_{n\geqslant 1} f_n x^{2n-1} + \sum_{n\geqslant 1} f_n x^{2n} + \sum_{n\geqslant 1} f_{n+1}x^{2n}$$

or equivalently

$$\sum_{n\geqslant 1} f_n x^{n-1} = 1 + \sum_{n\geqslant 1} f_n x^{2n-1} + \sum_{n\geqslant 1} f_n x^{2n} + \sum_{n\geqslant 1} f_{n+1}x^{2n}.$$

This exactly means that $F(x) = x^2F(x^2) + xF(x^2) + F(x^2)$ i.e.

$$F(x) = (1+x+x^2)F(x^2).$$

Moreover we have

$$F(x) = \prod_{i=0}^{\infty}\left(1 + x^{2^i} + x^{2^{i+1}}\right).$$

We can show that the sequence defined by the previous formula has an interesting property. For every positive integer n we perform the following procedure: Write n in a binary expansion, discard the last ”block” of zeroes (if it exists), and group the remaining digits in as few blocks as possible such that each block contains the digits of the same type. If for two numbers m and n the corresponding sets of blocks coincide the we have $f_m = f_n$. For example the binary expansion of 22 is 10110 hence the set of corresponding blocks is $\{1, 0, 11\}$, while the number 13 is represented as 1101 and has the very same set of blocks $\{11, 0, 1\}$, so we should have $f(22) = f(13)$. Easy verification gives us $f(22) = f(13) = 5$. From the last expression we conclude that f_n is the number of representations of n as a sum of powers of two, such that no two powers of two are taken from the same set of a collection $\{1,2\}$, $\{2,4\}$, $\{4,8\}$.

4 The Method of the Snake Oil

The method of the snake oil is very useful tool in evaluating various, frequently huge combinatorial sums, and in proving combinatorial identities.

The method is used to calculate many sums and as such it is not universal. Thus we will use several examples to give the flavor and illustration of the method.

The general principle is as follows: Suppose we want to calculate the sum S. First we wnat to identify the free variable on which S depends. Assume that n is such a variable and let $S = f(n)$. After that we have to obtain $F(x)$, the generating function of the sequence $f(n)$. We will multiply S by x^n and summ over all n. At this moment we have (at least) a double summation external in n and internal in S. Then we interchange the order of summation and get the value of internal sum in terms of n. In such a way we get certain coefficients of the generating function which are in fact the values of S in dependence of n.

In solving problems of this type we usually encounter several sums. Here we will first list some of these sums.

The identity involving $\sum_n \binom{m}{n} x^n$ is known from before:

$$(1+x)^m = \sum_n \binom{m}{n} x^n.$$

Sometimes we will use the identity for $\sum_n \binom{n}{k} x^n$ which is already mentioned in the list of generating functions:

$$\frac{1}{(1-x)^{k+1}} = \sum_n \binom{n+k}{k} x^n.$$

Among the common sums we will encounter those involving only even (or odd) indeces. For example we have $(1+x)^m = \sum_n \binom{m}{n} x^n$, hence $(1-x)^m = \sum_n \binom{m}{n} (-x)^n$. Adding and subtracting yields:

$$\sum_n \binom{m}{2n} x^{2n} = \frac{((1+x)^m + (1-x)^m)}{2},$$

$$\sum_n \binom{m}{2n+1} x^{2n+1} = \frac{((1+x)^m - (1-x)^m)}{2}.$$

In a similar fashion we prove:

$$\sum_n \binom{2n}{m} x^{2n} = \frac{x^m}{2}\left(\frac{1}{(1-x)^{m+1}} + \frac{(-1)^m}{(1-x)^{m+1}}\right), \text{ and}$$

$$\sum_n \binom{2n+1}{m} x^{2n+1} = \frac{x^m}{2}\left(\frac{1}{(1-x)^{m+1}} - \frac{(-1)^m}{(1-x)^{m+1}}\right).$$

The following identity is also used quite frequently:

$$\sum_n \frac{1}{n+1}\binom{2n}{n} x^n = \frac{1}{2x}(1-\sqrt{1-4x}).$$

Problem 9. *Evaluate the sum*

$$\sum_k \binom{k}{n-k}.$$

Solution. Let n be the free variable and denote the sum by

$$f(n) = \sum_k \binom{k}{n-k}.$$

Let $F(x)$ be the generating function of the sequence $f(n)$, i.e.

$$F(x) = \sum_n x^n f(n) = \sum_n x^n \sum_k \binom{k}{n-k} = \sum_n \sum_k \binom{k}{n-k} x^n.$$

We can rewrite the previous equation as

$$F(x) = \sum_k \sum_n \binom{k}{n-k} x^n = \sum_k x^k \sum_n \binom{k}{n-k} x^{n-k},$$

which gives

$$F(x) = \sum_k x^k (1+x)^k = \sum_k (x+x^2)^k = \frac{1}{1-(x-x^2)} = \frac{1}{1-x-x^2}.$$

However this is very similar to the generating function of a Fibonacci's sequence, i.e. $f(n) = F_{n+1}$ and we arrive to

$$\sum_k \binom{k}{n-k} = F_{n+1}. \; \triangle$$

Problem 10. *Evaluate the sum*

$$\sum_{k=m}^{n}(-1)^k\binom{n}{k}\binom{k}{m}.$$

Solution. If n is a fixed number, then m is a free variable on which the sum depends. Let $f(m) = \sum_{k=m}^{n}(-1)^k\binom{n}{k}\binom{k}{m}$ and let $F(x)$ be the generating function of the sequence $f(m)$, i.e. $F(x) = \sum_m f(m)x^m$. Then we have

$$F(x) = \sum_m f(m)x^m = \sum_m x^m \sum_{k=m}^{n}(-1)^k\binom{n}{k}\binom{k}{m} =$$

$$= \sum_{k\leqslant n}(-1)^k\binom{n}{k}\sum_{m\leqslant k}\binom{k}{m}x^m = \sum_{k\leqslant n}\binom{n}{k}(1+x)^k.$$

Here we have used $\sum_{m\leqslant k}\binom{k}{m}x^m = (1+x)^k$. Dalje je

$$F(x) = (-1)^n\sum_{k\leqslant n}\binom{n}{k}(-1)^{n-k}(1+x)^k = (-1)^n\Big((1+x)-1\Big)^n = (-1)^n x^n$$

Therefore we obtained $F(x) = (-1)^n x^n$ and since this is a generating function of the sequence $f(m)$ we have

$$f(m) = \begin{cases} (-1)^n, & n = m \\ 0, & m < n\,. \end{cases} \triangle$$

Problem 11. *Evaluate the sum* $\sum_{k=m}^{n}\binom{n}{k}\binom{k}{m}$.

Solution. Let $f(m) = \sum_{k=m}^{n}\binom{n}{k}\binom{k}{m}$ and $F(x) = \sum_m x^m f(m)$. Then we have

$$F(x) = \sum_m x^m f(m) = \sum_m x^m \sum_{k=m}^{n}\binom{n}{k}\binom{k}{m} = \sum_{k\leqslant n}\binom{n}{k}\sum_{m\leqslant k}\binom{k}{m}x^m = \sum_{k\leqslant n}\binom{n}{k}(1+x)^k,$$

implying $F(x) = (2+x)^n$. Since

$$(2+x)^n = \sum_m\binom{n}{m}2^{n-m}x^m,$$

the value of the required sum is $f(m) = \binom{n}{m}2^{n-m}$. $\triangle$

Problem 12. *Evaluate*

$$\sum_k\binom{n}{\left[\frac{k}{2}\right]}x^k.$$

Solution. We can divide this into two sums

$$\sum_k\binom{n}{\left[\frac{k}{2}\right]}x^k = \sum_{k=2k_1}\binom{n}{\left[\frac{2k_1}{2}\right]}x^{2k_1} + \sum_{k=2k_2+1}\binom{n}{\left[\frac{2k_2+1}{2}\right]}x^{2k_2+1} =$$

$$= \sum_{k_1}\binom{n}{k_1}(x^2)^{k_1} + x\sum_{k_2}\binom{n}{k_2}(x^2)^{k_2} = (1+x^2)^n + x(1+x^2)^n,$$

or equivalently

$$\sum_k\binom{n}{\left[\frac{k}{2}\right]}x^k = (1+x)(1+x^2)^n.\ \triangle$$

Problem 13. *Determine the elements of the sequence:*

$$f(m)=\sum_k\binom{n}{k}\binom{n-k}{\left[\frac{m-k}{2}\right]}y^k.$$

Solution. Let $F(x)=\sum_m x^m f(m)$. We then have

$$F(x)=\sum_m x^m\sum_k\binom{n}{k}\binom{n-k}{\left[\frac{m-k}{2}\right]}y^k=\sum_k\binom{n}{k}y^k\sum_m\binom{n-k}{\left[\frac{m-k}{2}\right]}x^m=$$

$$=\sum_k\binom{n}{k}y^kx^k\sum_m\binom{n-k}{\left[\frac{m-k}{2}\right]}x^{m-k}=\sum_k\binom{n}{k}y^kx^k(1+x)(1+x^2)^{n-k}.$$

Hence

$$F(x)=(1+x)\sum_k\binom{n}{k}(1+x^2)^{n-k}(xy)^k=(1+x)(1+x^2+xy)^n.$$

For $y=2$ we have that $F(x)=(1+x)^{2n+1}$, implying that $F(x)$ is the generating function of the sequence $\binom{2n+1}{m}$ and we get the following combinatorial identity:

$$\sum_k\binom{n}{k}\binom{n-k}{\left[\frac{m-k}{2}\right]}2^k=\binom{2n+1}{m}.$$

Setting $y=-2$ we get $F(x)=(1+x)(1-x)^{2n}=(1-x)^{2n}+x(1-x)^{2n}$ hence the coefficient near x^m equals $\binom{2n}{m}(-1)^m+\binom{2n}{m-1}(-1)^{m-1}=(-1)^m\left[\binom{2n}{m}-\binom{2n}{m-1}\right]$ which implies

$$\sum_k\binom{n}{k}\binom{n-k}{\left[\frac{m-k}{2}\right]}(-2)^k=(-1)^m\left[\binom{2n}{m}-\binom{2n}{m-1}\right].\ \triangle$$

Problem 14. *Prove that*

$$\sum_k\binom{n}{k}\binom{k}{j}x^k=\binom{n}{j}x^j(1+x)^{n-j}$$

for each $n\geqslant 0$

Solution. If we fix n and let j be the free variable and $f(j)=\sum_k\binom{n}{k}\binom{k}{j}x^k$, $g(j)=\binom{n}{j}x^j(1+x)^{n-j}$, then the corresponding generating functions are

$$F(y)=\sum_j y^jf(j),\qquad G(y)=\sum_j y^jg(j).$$

We want to prove that $F(y)=G(y)$. We have

$$F(y)=\sum_j y^j\sum_k\binom{n}{k}\binom{k}{j}x^k=\sum_k\binom{n}{k}x^k\sum_j\binom{k}{j}y^j=\sum_k\binom{n}{k}x^k(1+y)^k,$$

hence $F(y)=(1+x+xy)^n$. On the other hand we have

$$G(y)=\sum_j y^j\binom{n}{j}x^j(1+x)^{n-j}=\sum_j\binom{n}{j}(1+x)^{n-j}(xy)^j=(1+x+xy)^n,$$

hence $F(y)=G(y)$. $\triangle$

The real power of the generating functions method can be seen in the following example.

Problem 15. *Evaluate the sum*

$$\sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1}$$

for $m,n \geqslant 0$.

Solution. Since there are quite a lot of variables elementary combinatorial methods doesn't offer an effective way to treat the sum. Since n appears on only one place in the sum, it is natural to consider the sum as a function on n Let $F(x)$ be the generating series of such functions. Then

$$F(x)=\sum_n x^n \sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1}=\sum_k \binom{2k}{k}\frac{(-1)^k}{k+1}x^{-k}\sum_n \binom{n+k}{m+2k}x^{n+k}=$$

$$=\sum_k \binom{2k}{k}\frac{(-1)^k}{k+1}x^{-k}\frac{x^{m+2k}}{(1-x)^{m+2k+1}}=\frac{x^{m+2k}}{(1-x)^{m+2k+1}}\sum_k \binom{2k}{k}\frac{1}{k+1}\left\{\frac{-x}{(1-x)^2}\right\}^k=$$

$$=\frac{-x^{m-1}}{2(1-x)^{m-1}}\left\{1-\sqrt{1+\frac{4x}{(1-x)^2}}\right\}=\frac{-x^{m-1}}{2(1-x)^{m-1}}\left\{1-\frac{1+x}{1-x}\right\}=\frac{x^m}{(1-x)^m}.$$

This is a generating function of the sequence $\binom{n-1}{m-1}$ which establishes

$$\sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1}=\binom{n-1}{m-1}. \quad \triangle$$

Problem 16. *Prove the identity*

$$\sum_k \binom{2n+1}{k}\binom{m+k}{2n}=\binom{2m+1}{2n}.$$

Solution. Let $F(x)=\sum_m x^m \sum_k \binom{2n+1}{k}\binom{m+k}{2n}$ and $G(x)=\sum_m x^m \binom{2m+1}{2n}$ the generating functions of the expressions on the left and right side of the required equality. We will prove that $F(x)=G(x)$. We have

$$F(x)=\sum_m x^m \sum_k \binom{2n+1}{k}\binom{m+k}{2n}=\sum_k \binom{2n+1}{2k}\sum_m \binom{m+k}{2n}=$$

$$=\sum_k \binom{2n+1}{2k}\sum_m \binom{m+k}{2n}x^m=\sum_k \binom{2n+1}{2k}x^{-k}\sum_m \binom{m+k}{2n}x^{m+k}=$$

$$=\sum_k \binom{2n+1}{2k}x^{-k}\frac{x^{2n}}{(1-x)^{2n+1}}=\frac{x^{2n}}{(1-x)^{2n+1}}\sum_k \binom{2n+1}{2k}\left(x^{-\frac{1}{2}}\right)^{2k}.$$

We already know that $\sum_k \binom{2n+1}{2k}\left(x^{-\frac{1}{2}}\right)^{2k}=\frac{1}{2}\left(\left(1+\frac{1}{\sqrt{x}}\right)^{2n+1}+\left(1-\frac{1}{\sqrt{x}}\right)^{2n+1}\right)$ so

$$F(x)=\frac{1}{2}(\sqrt{x})^{2n-1}\left(\frac{1}{(1-\sqrt{x})^{2n+1}}-\frac{1}{(1+\sqrt{x})^{2n+1}}\right).$$

On the other hand

$$G(x)=\sum_m \binom{2m+1}{2n}x^m=\left(x^{-1/2}\right)\sum_m \binom{2m+1}{2n}\left(x^{1/2}\right)^{2m+1},$$

implying

$$G(x)=\left(x^{-1/2}\right)\left[\frac{(x^{1/2})^{2n}}{2}\left(\frac{1}{(1-x^{1/2})^{2n+1}}-(-1)^{2n}\frac{1}{(1+x^{1/2})^{2n+1}}\right)\right],$$

or

$$G(x)=\frac{1}{2}(\sqrt{x})^{2n-1}\left(\frac{1}{(1-\sqrt{x})^{2n+1}}-\frac{1}{(1+\sqrt{x})^{2n+1}}\right).\ \triangle$$

Problem 17. *Prove that*

$$\sum_{k=0}^{n}\binom{2n}{2k}\binom{2k}{k}2^{2n-2k}=\binom{4n}{2n}.$$

Let n be the free variable on the left and right side of $F(x)$ and $G(x)$. We want to prove the equality of these generating functions.

$$F(x)=\sum_n x^n\sum_{0\leqslant k\leqslant n}\binom{2n}{2k}\binom{2k}{k}2^{2n-2k}=\sum_{0\leqslant k}\binom{2k}{k}2^{-2k}\sum_n\binom{2n}{2k}x^n2^{2n},$$

$$F(x)=\sum_{0\leqslant k}\binom{2k}{k}2^{-2k}\sum_n\binom{2n}{2k}(2\sqrt{x})^{2n}.$$

Now we use the formula for summation of even powers and get

$$\sum_n\binom{2n}{2k}(2\sqrt{x})^{2n}=\frac{1}{2}(2\sqrt{x})^{2k}\left(\frac{1}{(1-2\sqrt{x})^{2k+1}}+\frac{1}{(1+2\sqrt{x})^{2k+1}}\right),$$

and we further get

$$F(x)=\frac{1}{2(1-2\sqrt{x})}\sum_k\binom{2k}{k}\left(\frac{x}{(1-2\sqrt{x})^2}\right)^k+\frac{1}{2(1+2\sqrt{x})}\sum_k\binom{2k}{k}\left(\frac{x}{(1+2\sqrt{x})^2}\right)^k.$$

Since $\sum_n\binom{2n}{n}x^n=\frac{1}{\sqrt{1-4x}}$ we get

$$F(x)=\frac{1}{2(1-2\sqrt{x})}\cdot\frac{1}{\sqrt{1-4\frac{x}{(1-2\sqrt{x})^2}}}+\frac{1}{2(1+2\sqrt{x})}\cdot\frac{1}{\sqrt{1-4\frac{x}{(1+2\sqrt{x})^2}}},$$

which implies

$$F(x)=\frac{1}{2\sqrt{1-4\sqrt{x}}}+\frac{1}{2\sqrt{1+4\sqrt{x}}}.$$

On the other hand for $G(x)$ we would like to get the sum $\sum_n\binom{4n}{2n}x^n$. Since $\sum_n\binom{2n}{n}x^n=\frac{1}{\sqrt{1-4x}}$ we have $\sum_n\binom{2n}{n}(-x)^n=\frac{1}{\sqrt{1+4x}}$ hence

$$G(x)=\frac{1}{2}\left(\frac{1}{\sqrt{1-4\sqrt{x}}}+\frac{1}{\sqrt{1+4\sqrt{x}}}\right)$$

and $F(x)=G(x)$. $\triangle$

The followng problem is slightly harder because the standard idea of snake oil doesn't lead to a solution.

Problem 18 (Moriati). *For given n and p evaluate*

$$\sum_k \binom{2n+1}{2p+2k+1}\binom{p+k}{k}.$$

Solution. In order to have shorter formulas let us introduce $r = p+k$. If we assume that n is the free variable then the required sum is equal to

$$f(n) = \sum_r \binom{2n+1}{2r+1}\binom{r}{p}.$$

Take $F(x) = \sum_n x^{2n+1} f(n)$. This is somehow natural since the binomial coefficient contains the term $2n+1$. Now we have

$$F(x) = \sum_n x^{2n+1} \sum_r \binom{2n+1}{2r+1}\binom{r}{p} = \sum_r \binom{r}{p} \sum_n \binom{2n+1}{2r+1} x^{2n+1}.$$

Since

$$\sum_n \binom{2n+1}{2r+1} x^{2n+1} = \frac{x^{2r+1}}{2}\left(\frac{1}{(1-x)^{2r+2}} + \frac{1}{(1+x)^{2r+2}}\right),$$

we get

$$F(x) = \frac{1}{2}\cdot\frac{x}{(1-x)^2}\sum_r \binom{r}{p}\left(\frac{x^2}{(1-x)^2}\right)^r + \frac{1}{2}\cdot\frac{x}{(1+x)^2}\sum_r \binom{r}{p}\left(\frac{x^2}{(1+x)^2}\right)^r,$$

$$F(x) = \frac{1}{2}\frac{x}{(1-x)^2}\frac{\left(\frac{x^2}{(1-x)^2}\right)^p}{\left(1-\frac{x^2}{(1-x)^2}\right)^{p+1}} + \frac{1}{2}\frac{x}{(1+x)^2}\frac{\left(\frac{x^2}{(1+x)^2}\right)^p}{\left(1-\frac{x^2}{(1+x)^2}\right)^{p+1}},$$

$$F(x) = \frac{1}{2}\frac{x^{2p+1}}{(1-2x)^{p+1}} + \frac{1}{2}\frac{x^{2p+1}}{(1+2x)^{p+1}} = \frac{x^{2p+1}}{2}((1+2x)^{-p-1} + (1-2x)^{-p-1}),$$

implying

$$f(n) = \frac{1}{2}\left(\binom{-p-1}{2n-2p}2^{2n-2p} + \binom{-p-1}{2n-2p}2^{2n-2p}\right),$$

and after simplification

$$f(n) = \binom{2n-p}{2n-2p}2^{2n-2p}. \ \triangle$$

We notice that for most of the problems we didn't make a substantial deviation from the method and we used only a handful of identities. This method can also be used in writing computer algorithms for symbolic evaluation of number of sums with binomial coefficients.

5 Problems

1. Prove that for the sequence of Fibonacci numbers we have

$$F_0 + F_1 + \cdots + F_n = F_{n+2} + 1.$$

2. Given a positive integer n, let A denote the number of ways in which n can be partitioned as a sum of odd integers. Let B be the number of ways in which n can be partitioned as a sum of different integers. Prove that $A = B$.

3. Find the number of permutations without fixed points of the set $\{1, 2, \ldots, n\}$.

4. Evaluate $\sum_k (-1)^k \binom{n}{3k}$.

5. Let $n \in \mathbb{N}$ and assume that

$$\begin{aligned} x+2y=n &\quad \text{has } R_1 \text{ solutions in } \mathbb{N}_0^2 \\ 2x+3y=n-1 &\quad \text{has } R_2 \text{ solutions in } \mathbb{N}_0^2 \\ &\vdots \\ nx+(n+1)y=1 &\quad \text{has } R_n \text{ solutions in } \mathbb{N}_0^2 \\ (n+1)x+(n+2)y=0 &\quad \text{has } R_{n+1} \text{ solutions in } \mathbb{N}_0^2 \end{aligned}$$

Prove that $\sum_k R_k = n+1$.

6. A polynomial $f(x_1,x_2,\dots,x_n)$ is called a *symmetric* if each permutation $\sigma \in S_n$ we have $f(x_{\sigma(1)},\dots,x_{\sigma(n)}) = f(x_1,\dots,x_n)$. We will consider several classes of symmetric polynomials. The first class consists of the polynomials of the form:

$$\sigma_k(x_1,\dots,x_n) = \sum_{i_1<\dots<i_k} x_{i_1}\cdot\dots\cdot x_{i_k}$$

for $1 \leqslant k \leqslant n$, $\sigma_0 = 1$, and $\sigma_k = 0$ for $k > n$. Another class of symmetric polynomials are the polynomials of the form

$$p_k(x_1,\dots,x_n) = \sum_{i_1+\dots+i_n=k} x_1^{i_1}\cdot\dots\cdot x_n^{i_n}, \qquad \text{where } i_1,\cdots,i_n \in \mathbb{N}_0.$$

The third class consists of the polynomials of the form:

$$s_k(x_1,\dots,x_n) = x_1^k + \dots + x_n^k.$$

Prove the following relations between the polynomials introduced above:

$$\sum_{r=0}^{n}(-1)^r\sigma_r p_{n-r} = 0,\ np_n = \sum_{r=1}^{n} s_r p_{n-r},\ \text{and } n\sigma_n = \sum_{r=1}^{n}(-1)^{r-1}s_r\sigma_{n-r}.$$

7. Assume that for some $n \in \mathbb{N}$ there are sequences of positive numbers $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ such that the sums

$$a_1+a_2,\ a_1+a_3,\ \dots,\ a_{n-1}+a_n$$

and

$$b_1+b_2,\ b_1+b_3,\ \dots,\ b_{n-1}+b_n$$

the same up to permutation. Prove that n is a power of two.

8. (Leo Moser, Joe Lambek, 1959.) Prove that there is a unique way to partition the set of natural numbers in two sets A and B such that: For very non-negative integer n (including 0) the number of ways in which n can be written as $a_1+a_2,\ a_1,a_2 \in A,\ a_1 \neq a_2$ is at least 1 and is equal to the number of ways in which it can be represented as $b_1+b_2,\ b_1,b_2 \in B,\ b_1 \neq b_2$.

9. Given several (at least two, but finitely many) arithmetic progressions, if each natural number belongs to exactly one of them, prove there are two progressions whose common differences are equal.

10. (This problem was posed in the journal *American Mathematical Monthly*) Prove that in the contemporary calendar the $13th$ in a month is most likely to be Friday.

    Remark: The contemporary calendar has a period of 400 years. Every fourth year has 366 days except those divisible by 100 and not by 400.

6 Solultions

1. According to the Theorem 7 the generating function of the sum of first n terms of the sequence (i.e. the left-hand side) is equal to $F/(1-x)$, where $F = x/(1-x-x^2)$ (such F is the generatring function of the Fibonacci sequence). On the right-hand side we have

$$\frac{F-x}{x} - \frac{1}{1-x},$$

and after some obvious calculation we arrive to the required identity.

2. We will first prove that the generating function of the number of odd partitions is equal to

$$(1+x+x^2+\cdots)\cdot(1+x^3+x^6+\cdots)\cdot(1+x^5+x^{10}+\cdots)\cdots = \prod_{k\geqslant 1}\frac{1}{1-x^{2k+1}}.$$

Indeed, to each partition in which i occurs a_i times corresponds exactly one term with coefficient 1 in the product. That term is equal to $x^{1\cdot a_1+3\cdot a_3+5\cdot a_5+\cdots}$.

The generating function to the number of partitions in different summands is equal to

$$(1+x)\cdot(1+x^2)\cdot(1+x^3)\cdots = \prod_{k\geqslant 1}(1+x^k),$$

because from each factor we may or may not take a power of x, which exactly correpsonds to taking or not taking the corresponding summand of a partition. By some elementary transformations we get

$$\prod_{k\geqslant 1}(1+x^k) = \prod_{k\geqslant 1}\frac{1-x^{2k}}{1-x^k} = \frac{(1-x^2)(1-x^4)\cdots}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots} = \prod_{k\geqslant 1}\frac{1}{1-x^{2k+1}}$$

which proves the statement.

3. This example illustrates the usefullness of the exponential generating functions. This problem is known as *derangement problem* or ”le Problème des Rencontres” posed by Pierre R. de Montmort (1678-1719).

Assume that the required number is D_n and let $D(x) \overset{esr}{\leftrightarrow} D_n$. The number of permutations having exactly k given fixed points is equal to D_{n-k}, hence the total number of permutations with exactly k fixed points is equal to $\binom{n}{k}D_{n-k}$, because we can choose k fixed points in $\binom{n}{k}$ ways. Since the total number of permutations is equal to $n!$, then

$$n! = \sum_k \binom{n}{k} D_{n-k}$$

and the Theorem 10 gives

$$\frac{1}{1-x} = e^x D(x)$$

implying $D(x) = e^{-x}/(1-x)$. Since e^{-x} is the generating function of the sequence $\frac{(-1)^n}{n!}$, we get

$$\frac{D_n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n\frac{1}{n!},$$

$$D_n = n!\cdot\left(\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n\frac{1}{n!}\right).$$

4. The idea here is to consider the generating function

$$F(x)=\sum_k \binom{n}{3k} x^{3k}.$$

The required sum is equal to $f(-1)$. The question now is how to make binomial formula to skip all terms except those of order $3k$. We will use the following identy for the sum of roots of unity in the complex plane

$$\sum_{\varepsilon^r=1} \varepsilon^n = \begin{cases} r, & r|n \\ 0, & \text{otherwise.} \end{cases}$$

Let $C(x)=(1+x)^n$ and let 1, ε, and ε^2 be the cube roots of 1. Then we have

$$F(x)=\frac{C(x)+C(\varepsilon x)+C(\varepsilon^2 x)}{3}$$

which for $x=-1$ gives

$$F(-1)=\frac{1}{3}\left\{\left(\frac{3-i\sqrt{3}}{2}\right)^n+\left(\frac{3+i\sqrt{3}}{2}\right)^n\right\}$$

and after simplification

$$\sum_k (-1)^k \binom{n}{3k} = 2\cdot 3^{n/2-1}\cos\left(\frac{n\pi}{6}\right)$$

5. The number of solutions of $x+2y=n$ in $\mathbb{N}_0^2$ is the coefficient near t^n in

$$(1+t+t^2+\cdots)\cdot(1+t^2+t^4+\cdots)=\frac{1}{1-t}\frac{1}{1-t^2}$$

The reason is that each pair (x,y) that satisfies the condition of the problem increases the coefficient near t^n by 1 because it appears as a summand of the form $t^x t^{2y}=t^{x+2y}$. More generally, the number of solutions of $kx+(k+1)y=n+1-k$ is the coefficient near t^{n+1-k} in $\frac{1}{1-t^k}\frac{1}{1-t^{k+1}}$, i.e. the coefficient near t^n in $\frac{x^{k-1}}{(1-t^k)(1-t^{k+1})}$. Hence, $\sum_{k=1}^n R_k$ is the coefficient near t^n in $\sum_k \frac{t^{k-1}}{(1-t^k)(1-t^{k+1})}=\sum_k \frac{1}{t-t^2}\left(\frac{1}{1-t^{k+2}}-\frac{1}{1-t^{k+1}}\right)=\frac{1}{(1-t)^2}$. Now it is easy to see that $\sum_k R_k=n+1$.

6. The generating function of the symmetric polynomials $\sigma_k(x_1,\dots,x_n)$ is

$$\Sigma(t)=\sum_{k=0}^{\infty}\sigma_k t^k=\prod_{i=1}^{n}(1+tx_i).$$

The generating function of the polynomials $p_k(x_1,\dots,x_n)$ is:

$$P(t)=\sum_{k=0}^{\infty}p_k t^k=\prod\frac{1}{1-tx_i},$$

and the generating function of the polynomials s_k is:

$$S(t)=\sum_{k=0}^{\infty}s_k t^{k-1}=\sum_{i=1}^{n}\frac{x_i}{1-tx_i}.$$

The functions $\Sigma(t)$ and $P(t)$ satisfy the following condition $\Sigma(t)p(-t) = 1$. If we calculate the coefficient of this product near t^n, $n \geqslant 1$ we get the relation

$$\sum_{r=0}^{n} (-1)^r \sigma_r p_{n-r} = 0.$$

Notice that

$$\log P(t) = \sum_{i=1}^{n} \log \frac{1}{1-tx_i} \qquad \text{and} \qquad \log \Sigma(t) = \sum_{i=1}^{n} \log(1+tx_i).$$

Now we can express $S(t)$ in terms of $P(t)$ and $\Sigma(t)$ by:

$$S(t) = \frac{d}{dt} \log P(t) = \frac{P'(t)}{P(t)}$$

and

$$S(-t) = -\frac{d}{dt} \log \Sigma(t) = -\frac{\Sigma'(t)}{\Sigma(t)}.$$

From the first formula we get $S(t)P(t) = P'(t)$, and from the second $S(-t)\Sigma(t) = -\Sigma'(t)$. Comparing the coefficients near t^{n+1} we get

$$np_n = \sum_{r=1}^{n} s_r p_{n-r} \qquad \text{and} \qquad n\sigma_n = \sum_{r=1}^{n} (-1)^{r-1} s_r \sigma_{n-r}.$$

7. Let F and G be polynomials generated by the given sequence: $F(x) = x^{a_1} + x^{a_2} + \cdots + x^{a_n}$ and $G(x) = x^{b_1} + x^{b_2} + \cdots + x^{b_n}$. Then

$$\begin{aligned} F^2(x) - G^2(x) &= \left(\sum_{i=1}^{n} x^{2a_i} + 2 \sum_{1 \leqslant i \leqslant j \leqslant n} x^{a_i + a_j} \right) - \left(\sum_{i=1}^{n} x^{2b_i} + 2 \sum_{1 \leqslant i \leqslant j \leqslant n} x^{b_i + b_j} \right) \\ &= F(x^2) - G(x^2). \end{aligned}$$

Since $F(1) = G(1) = n$, we have that 1 is zero of the order $k, (k \geqslant 1)$ of the polynomial $F(x) - G(x)$. Then we have $F(x) - G(x) = (x-1)^k H(x)$, hence

$$F(x) + G(x) = \frac{F^2(x) - G^2(x)}{F(x) - G(x)} = \frac{F(x^2) - G(x^2)}{F(x) - G(x)} = \frac{(x^2-1)^k H(x^2)}{(x-1)^k H(x)} = (x+1)^k \frac{H(x^2)}{H(x)}$$

Now for $x = 1$ we have:

$$2n = F(1) + G(1) = (1+1)^k \frac{H(x^2)}{H(x)} = 2^k,$$

implying that $n = 2^{k-1}$.

8. Consider the polynomials generated by the numbers from different sets:

$$A(x) = \sum_{a \in A} x^a, \quad B(x) = \sum_{b \in B} x^b.$$

The condition that A and B partition the whole $\mathbb{N}$ without intersection is equivalent to

$$A(x) + B(x) = \frac{1}{1-x}.$$

The number of ways in which some number can be represented as $a_1+a_2,\ \ a_1,a_2\in A,\ \ a_1\neq a_2$ has the generating function:

$$\sum_{a_i,a_j\in A,a_i\neq a_j} x^{a_i+a_j}=\frac{1}{2}\left(A^2(x)-A(x^2)\right).$$

Now the second condition can be expressed as

$$\left(A^2(x)-A(x^2)\right)=\left(B^2(x)-B(x^2)\right).$$

We further have

$$(A(x)-B(x))\frac{1}{1-x}=A(x^2)-B(x^2)$$

or equivalently

$$(A(x)-B(x))=(1-x)(A(x^2)-B(x^2)).$$

Changing x by $x^2,x^4,\dots,x^{2^{n-1}}$ we get

$$A(x)-B(x)=(A(x^{2^n})-B(x^{2^n}))\prod_{i=0}^{n-1}(1-x^{2^i}),$$

implying

$$A(x)-B(x)=\prod_{i=0}^{\infty}(1-x^{2^i}).$$

The last product is series whose coefficients are ± 1 hence A and B are uniquely determined (since their coefficients are 1). It is not difficult to notice that positive coefficients (i.e. coefficients originating from A) are precisely those corresponding to the terms x^n for which n can be represented as a sum of even numbers of 2s. This means that the binary partition of n has an even number of 1s. The other numbers form B.

Remark: The sequence representing the parity of the number of ones in the binary representation of n is called *Morse* sequence.

9. This problem is posed by Erdösz (in slightly different form), and was solved by Mirsky and Newman after many years. This is their original proof:

   Assume that k arithmetic progressionss $\{a_i+nb_i\}\ \ (i=1,2,\dots,k)$ cover the entire set of positive integers. Then $\dfrac{z^a}{1-z^b}=\sum_{i=0}^{\infty}z^{a+ib}$, hence

$$\frac{z}{1-z}=\frac{z^{a_1}}{1-z^{b_1}}+\frac{z^{a_2}}{1-z^{b_2}}+\cdots+\frac{z^{a_k}}{1-z^{b_k}}.$$

   Let $|z|\leqslant 1$. We will prove that the biggest number among b_i can't be unique. Assume the contrary, that b_1 is the greatest among the numbers $b_1,b_2,\dots,b_n$ and set $\varepsilon=e^{2i\pi/b_1}$. Assume that z approaches ε in such a way that $|z|\leqslant 1$. Here we can choose ε such that $\varepsilon^{b_1}=1$, $\varepsilon\neq 1$, and $\varepsilon^{b_i}\neq 1,\ \ 1<i\leqslant k$. All terms except the first one converge to certain number while the first converges to ∞, which is impossible.

10. Friday the 13th corresponds to Sunday the 1st. Denote the days by numbers $1,2,3,\dots$ and let t^i corresponds to the day i. Hence, $Jan.1st2001$ is denoted by 1 (or t), $Jan.4th2001$ by t^4 etc. Let A be the set of all days (i.e. corresponding numbers) which happen to be the first in a month. For instance, $1\in A$, $2\in A$, etc. $A=\{1,32,60,\dots\}$. Let $f_A(t)=\sum_{n\in A}t^n$. If we replace t^{7k} by 1, t^{7k+1} by t, t^{7k+2} by t^2 etc. in the polynomial f_A we get another polynomial – denote it by $g_A(t)=\sum_{i=0}^{6}a_it^i$. Now the number a_i represents how many times the day (of a week)

denoted by i has appeared as the first in a month. Since $Jan1,2001$ was Monday, a_1 is the number of Mondays, a_2- the number of Tuesdays, ..., a_0- the number of Sundays. We will consider now f_A modulus t^7-1. The polynoimal $f_A(t)-g_A(t)$ is divisible by t^7-1. Since we only want to find which of the numbers $a_0,a_1,\dots,a_6$ is the biggest, it is enough to consider the polynomial modulus $q(t)=1+t+t^2+\cdots+t^6$ which is a factor of t^7-1. Let $f_1(t)$ be the polynomial that represents the first days of months in 2001. Since the first day of January is Monday, Thursday– the first day of February, ..., Saturday the first day of December, we get

$$f_1(t)=t+t^4+t^4+1+t^2+t^5+1+t^3+t^6+t+t^4+t^6=$$

$$=2+2t+t^2+t^3+3t^4+t^5+2t^6\equiv 1+t+2t^4+t^6 \pmod{q(t)}.$$

Since the common year has $365\equiv 1 \pmod 7$ days, polynomials $f_2(t)$ and $f_3(t)$ corresponding to 2002. and 2003., satisfy

$$f_2(t)\equiv tf_1(t)\equiv tg_1(t)$$

and

$$f_3(t)\equiv tf_2(t)\equiv t^2g_1(t),$$

where the congruences are modulus $q(t)$. Using plain counting we easily verify that $f_4(t)$ for leap 2004 is

$$f_4(t)=2+2t+t^2+2t^3+3t^4+t^5+t^6\equiv 1+t+t^3+2t^4=g_4(t).$$

We will introduce a new polynomial that will count the first days for the period $2001-2004$ $h_1(t)=g_1(t)(1+t+t^2)+g_4(t)$. Also, after each common year the days are shifted by one place, and after each leap year by 2 places, hence after the period of 4 years all days are shifted by 5 places. In such a way we get a polynomial that counts the numbers of first days of months between 2001 and 2100. It is:

$$p_1(t)=h_1(t)(1+t^5+t^{10}+\cdots+t^{115})+t^{120}g_1(t)(1+t+t^2+t^3).$$

Here we had to write the last for years in the form $g_1(t)(1+t+t^2+t^3)$ because 2100 is not leap, and we can't replace it by $h_1(t)$. The period of 100 years shifts the calendar for 100 days (common years) and additional 24 days (leap) which is congruent to 5 modulus 7. Now we get

$$g_A(t)\equiv p_1(t)(1+t^5+t^{10})+t^{15}h_1(t)(1+t^5+\cdots+t^{120}).$$

Similarly as before the last 100 are counted by last summands because 2400 is leap. Now we will use that $t^{5a}+t^{5(a+1)}+\cdots+t^{5(a+6)}\equiv 0$. Thus $1+t^5+\cdots+t^{23\cdot 5}\equiv 1+t^5+t^{2\cdot 5}\equiv 1+t^3+t^5$ and $1+t^5+\cdots+t^{25\cdot 5}\equiv 1+t^5+t^{2\cdot 5}+t^{4\cdot 5}\equiv 1+t+t^3+t^5$. We further have that

$$p_1(t)\equiv h_1(t)(1+t^3+t^5)+tg_1(t)(1+t+t^2+t^3)\equiv$$

$$g_1(t)[(1+t+t^2)(1+t^3+t^5)+t(1+t+t^2+t^3)]+g_4(t)(1+t^3+t^5)\equiv$$

$$g_1(t)(2+2t+2t^2+2t^3+2t^4+2t^5+t^6)+g_4(t)(1+t^3+t^5)\equiv -g_1(t)t^6+g_4(t)(1+t^3+t^5).$$

If we now put this into formula for $g_A(t)$ we get

$$\begin{aligned}
g_A(t) &\equiv p_1(t)(1+t^3+t^5)+th_1(t)(1+t+t^3+t^5)\\
&\equiv -g_1(t)t^6(1+t^3+t^5)+g_4(t)(1+t^3+t^5)^2\\
&\quad +tg_1(t)(1+t+t^2)(1+t+t^3+t^5)+tg_4(t)(1+t+t^3+t^5)\\
&\equiv g_1(t)(t+t^3)+g_4(t)(2t+2t^3+t^5+t^6)\\
&\equiv (1+t+2t^4+t^6)(t+t^3)+(1+t+t^3+2t^4)(2t+2t^3+t^5+t^6)\\
&\equiv 8+4t+7t^2+5t^3+5t^4+7t^5+4t^6\equiv 4+3t^2+t^3+t^4+3t^5.
\end{aligned}$$

This means that the most probable day for the first in a month is Sunday (because a_0 is the biggest).

We can precisely determine the probability. If we use the fact that there are 4800 months in a period of 400, we can easily get the Sunday is the first exactly 688 times, Monday – 684, Tuesday – 687, Wednesday – 685, Thursday – 685, Friday – 687, and Saturday – 684.

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Complex Numbers in Geometry

Marko Radovanović
radmarko@yahoo.com

Contents

1 Introduction

When we are unable to solve some problem in plane geometry, it is recommended to try to do calculus. There are several techniques for doing calculations instead of geometry. The next text is devoted to one of them – the application of complex numbers.

The plane will be the complex plane and each point has its corresponding complex number. Because of that points will be often denoted by lowercase letters a, b, c, d, $\dots$, as complex numbers.

The following formulas can be derived easily.

2 Formulas and Theorems

Theorem 1.

- *$ab \parallel cd$ if and only if* $\dfrac{a-b}{\overline{a}-\overline{b}} = \dfrac{c-d}{\overline{c}-\overline{d}}$.
- *a,b,c are colinear if and only if* $\dfrac{a-b}{\overline{a}-\overline{b}} = \dfrac{a-c}{\overline{a}-\overline{c}}$.
- *$ab \perp cd$ if and only if* $\dfrac{a-b}{\overline{a}-\overline{b}} = -\dfrac{c-d}{\overline{c}-\overline{d}}$.
- *$\varphi = \angle acb$ (from a to b in positive direction) if and only if* $\dfrac{c-b}{|c-b|} = e^{i\varphi}\dfrac{c-a}{|c-a|}$.

Theorem 2. *Properties of the unit circle:*

- *For a chord ab we have* $\dfrac{a-b}{\overline{a}-\overline{b}} = -ab$.
- *If c belongs to the chord ab then* $\overline{c} = \dfrac{a+b-c}{ab}$.
- *The intersection of the tangents from a and b is the point* $\dfrac{2ab}{a+b}$.
- *The foot of perpendicular from an arbitrary point c to the chord ab is the point* $p = \dfrac{1}{2}\Big(a+b+c-ab\overline{c}\Big)$.
- *The intersection of chords ab and cd is the point* $\dfrac{ab(c+d)-cd(a+b)}{ab-cd}$.

Theorem 3. *The points* a,b,c,d *belong to a circle if and only if*

$$\frac{a-c}{b-c} : \frac{a-d}{b-d} \in \boldsymbol{R}.$$

Theorem 4. *The triangles abc and pqr are similar and equally oriented if and only if*

$$\frac{a-c}{b-c} = \frac{p-r}{q-r}.$$

Theorem 5. *The area of the triangle abc is*

$$p = \frac{i}{4}\begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix} = \frac{i}{4}\Big(a\overline{b}+b\overline{c}+c\overline{a}-\overline{a}b-\overline{b}c-\overline{c}a.\Big)$$

Theorem 6.
- *The point c divides the segment ab in the ratio* $\lambda \neq -1$ *if and only if* $c = \dfrac{a+\lambda b}{1+\lambda}$.
- *The point t is the centroid of the triangle abc if and only if* $t = \dfrac{a+b+c}{3}$.
- *For the orthocenter h and the circumcenter o of the triangle abc we have* $h+2o = a+b+c$.

Theorem 7. *Suppose that the unit circle is inscribed in a triangle abc and that it touches the sides* bc, ca, ab, *respectively at* p,q,r.

- *It holds* $a = \dfrac{2qr}{q+r}, b = \dfrac{2rp}{r+p}$ *and* $c = \dfrac{2pq}{p+q}$;
- *For the orthocenter h of the triangle abc it holds*

$$h = \frac{2(p^2q^2+q^2r^2+r^2p^2+pqr(p+q+r))}{(p+q)(q+r)(r+p)}.$$

- *For the excenter o of the triangle abc it holds* $o = \dfrac{2pqr(p+q+r)}{(p+q)(q+r)(r+p)}$.

Theorem 8.
- *For each triangle abc inscribed in a unit circle there are numbers* u,v,w *such that* $a = u^2, b = v^2, c = w^2$, *and* $-uv, -vw, -wu$ *are the midpoints of the arcs* ab, bc, ca *(respectively) that don't contain* c,a,b.
- *For the above mentioned triangle and its incenter i we have* $i = -(uv+vw+wu)$.

Theorem 9. *Consider the triangle $\triangle$ whose one vertex is 0, and the remaining two are x and y.*

- *If h is the orthocenter of $\triangle$ then* $h = \dfrac{(\overline{x}y + x\overline{y})(x-y)}{x\overline{y} - \overline{x}y}$.
- *If o is the circumcenter of $\triangle$, then* $o = \dfrac{xy(\overline{x} - \overline{y})}{\overline{x}y - x\overline{y}}$.

3 Complex Numbers and Vectors. Rotation

This section contains the problems that use the main properties of the interpretation of complex numbers as vectors (Theorem 6) and consequences of the last part of theorem 1. Namely, if the point b is obtained by rotation of the point a around c for the angle φ (in the positive direction), then $b - c = e^{i\varphi}(a - c)$.

1. (Yug MO 1990, 3-4 grade) Let S be the circumcenter and H the orthocenter of $\triangle ABC$. Let Q be the point such that S bisects HQ and denote by T_1, T_2, and T_3, respectively, the centroids of $\triangle BCQ$, $\triangle CAQ$ and $\triangle ABQ$. Prove that

$$AT_1 = BT_2 = CT_3 = \frac{4}{3}R,$$

where R denotes the circumradius of $\triangle ABC$.

2. (BMO 1984) Let $ABCD$ be an inscribed quadrilateral and let H_A, H_B, H_C and H_D be the orthocenters of the triangles BCD, CDA, DAB, and ABC respectively. Prove that the quadrilaterals $ABCD$ and $H_AH_BH_CH_D$ are congruent.

3. (Yug TST 1992) The squares $BCDE$, $CAFG$, and $ABHI$ are constructed outside the triangle ABC. Let $GCDQ$ and $EBHP$ be parallelograms. Prove that $\triangle APQ$ is isosceles and rectangular.

4. (Yug MO 1993, 3-4 grade) The equilateral triangles BCB_1, CDC_1, and DAD_1 are constructed outside the triangle ABC. If P and Q are respectively the midpoints of B_1C_1 and C_1D_1 and if R is the midpoint of AB, prove that $\triangle PQR$ is isosceles.

5. In the plane of the triangle $A_1A_2A_3$ the point P_0 is given. Denote with $A_s = A_{s-3}$, for every natural number $s > 3$. The sequence of points $P_0, P_1, P_2, \dots$ is constructed in such a way that the point P_{k+1} is obtained by the rotation of the point P_k for an angle 120^o in the clockwise direction around the point A_{k+1}. Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ has to be isosceles.

6. (IMO Shortlist 1992) Let $ABCD$ be a convex quadrilateral for which $AC = BD$. Equilateral triangles are constructed on the sides of the quadrilateral. Let O_1, O_2, O_3, and O_4 be the centers of the triangles constructed on AB, BC, CD, and DA respectively. Prove that the lines O_1O_3 and O_2O_4 are perpendicular.

4 The Distance. Regular Polygons

In this section we will use the following basic relation for complex numbers: $|a|^2 = a\overline{a}$. Similarly, for calculating the sums of distances it is of great advantage if points are colinear or on mutually parallel lines. Hence it is often very useful to use rotations that will move some points in nice positions.

Now we will consider the regular polygons. It is well-known that the equation $x^n = 1$ has exactly n solutions in complex numbers and they are of the form $x_k = e^{i\frac{2k\pi}{n}}$, for $0 \le k \le n-1$. Now we have that $x_0 = 1$ and $x_k = \varepsilon^k$, for $1 \le k \le n-1$, where $x_1 = \varepsilon$.

Let's look at the following example for the illustration:

Problem 1. *Let $A_0A_1A_2A_3A_4A_5A_6$ be a regular 7-gon. Prove that*

$$\frac{1}{A_0A_1} = \frac{1}{A_0A_2} + \frac{1}{A_0A_3}.$$

Solution. As mentioned above let's take $a_k = \varepsilon^k$, for $0 \le k \le 6$, where $\varepsilon = e^{i\frac{2\pi}{7}}$. Further, by rotation around $a_0 = 1$ for the angle ε, i.e. $\omega = e^{i\frac{2\pi}{14}}$, the points a_1 and a_2 are mapped to a_1' and a_2' respectively. These two points are collinear with a_3. Now it is enough to prove that $\frac{1}{a_1'-1} = \frac{1}{a_2'-1} + \frac{1}{a_3-1}$. Since $\varepsilon = \omega^2$, $a_1' = \varepsilon(a_1-1)+1$, and $a_2' = \omega(a_2-1)+1$ it is enough to prove that

$$\frac{1}{\omega^2(\omega^2-1)} = \frac{1}{\omega(\omega^4-1)} + \frac{1}{\omega^6-1}.$$

After rearranging we get $\omega^6+\omega^4+\omega^2+1 = \omega^5+\omega^3+\omega$. From $\omega^5 = -\omega^{12}$, $\omega^3 = -\omega^{10}$, and $\omega = -\omega^8$ (which can be easily seen from the unit circle), the equality follows from $0 = \omega^{12}+\omega^{10}+\omega^8+\omega^6+\omega^4+\omega^2+1 = \varepsilon^6+\varepsilon^5+\varepsilon^4+\varepsilon^3+\varepsilon^2+\varepsilon+1 = \frac{\varepsilon^7-1}{\varepsilon-1} = 0$. △

7. Let $A_0A_1\dots A_{14}$ be a regular 15-gon. Prove that

$$\frac{1}{A_0A_1} = \frac{1}{A_0A_2} + \frac{1}{A_0A_4} + \frac{1}{A_0A_7}.$$

8. Let $A_0A_1\dots A_{n-1}$ be a regular n-gon inscribed in a circle with radius r. Prove that for every point P of the circle and every natural number $m<n$ we have

$$\sum_{k=0}^{n-1} PA_k^{2m} = \binom{2m}{m} nr^{2m}.$$

9. (SMN TST 2003) Let M and N be two different points in the plane of the triangle ABC such that

$$AM:BM:CM = AN:BN:CN.$$

Prove that the line MN contains the circumcenter of $\triangle ABC$.

10. Let P be an arbitrary point on the shorter arc A_0A_{n-1} of the circle circumscribed about the regular polygon $A_0A_1\dots A_{n-1}$. Let $h_1, h_2, \dots, h_n$ be the distances of P from the lines that contain the edges A_0A_1, A_1A_2, ..., $A_{n-1}A_0$ respectively. Prove that

$$\frac{1}{h_1} + \frac{1}{h_2} + \cdots + \frac{1}{h_{n-1}} = \frac{1}{h_n}.$$

5 Polygons Inscribed in Circle

In the problems where the polygon is inscribed in the circle, it is often useful to assume that the unit circle is the circumcircle of the polygon. In theorem 2 we can see lot of advantages of the unit circle (especially the first statement) and in practice we will see that lot of the problems can be solved using this method. In particular, we know that each triangle is inscribed in the circle and in many problems from the geometry of triangle we can make use of complex numbers. The only problem in this task is finding the circumcenter. For that you should take a look in the next two sections.

11. The quadrilateral $ABCD$ is inscribed in the circle with diameter AC. The lines AB and CD intersect at M and the tangets to the circle at B and C interset at N. Prove that $MN \perp AC$.

12. (IMO Shorlist 1996) Let H be the orthocenter of the triangle $\triangle ABC$ and P an arbitrary point of its circumcircle. Let E the foot of perpendicular BH and let $PAQB$ and $PARC$ be parallelograms. If AQ and HR intersect in X prove that $EX \| AP$.

13. Given a cyclic quadrilateral $ABCD$, denote by P and Q the points symmetric to C with respect to AB and AD respectively. Prove that the line PQ passes through the orthocenter of $\triangle ABD$.

14. (IMO Shortlist 1998) Let ABC be a triangle, H its orthocenter, O its incenter, and R the circumradius. Let D be the point symmetric to A with respect to BC, E the point symmetric to B with respect to CA, and F the point symmetric to C with respect to AB. Prove that the points D, E, and F are collinear if and only if $OH = 2R$.

15. (Rehearsal Competition in MG 2004) Given a triangle ABC, let the tangent at A to the circumscribed circle intersect the midsegment parallel to BC at the point A_1. Similarly we define the points B_1 and C_1. Prove that the points A_1, B_1, C_1 lie on a line which is parallel to the Euler line of $\triangle ABC$.

16. (MOP 1995) Let AA_1 and BB_1 be the altitudes of $\triangle ABC$ and let $AB \neq AC$. If M is the midpoint of BC, H the orthocenter of $\triangle ABC$, and D the intersection of BC and B_1C_1, prove that $DH \perp AM$.

17. (IMO Shortlist 1996) Let ABC be an acute-angled triangle such that $BC > CA$. Let O be the circumcircle, H the orthocenter, and F the foot of perpendicular CH. If the perpendicular from F to OF intersects CA at P, prove that $\angle FHP = \angle BAC$.

18. (Romania 2005) Let $A_0A_1A_2A_3A_4A_5$ be a convex hexagon inscribed in a circle. Let A_0', A_2', A_4' be the points on that circle such that

$$A_0A_0' \parallel A_2A_4, \quad A_2A_2' \parallel A_4A_0 \quad A_4A_4' \parallel A_2A_0.$$

Suppose that the lines $A_0'A_3$ and A_2A_4 intersect at A_3', the lines $A_2'A_5$ and A_0A_4 intersect at A_5', and the lines $A_4'A_1$ and A_0A_2 intersect at A_1'.
If the lines A_0A_3, A_1A_4, and A_2A_5 are concurrent, prove that the lines A_0A_3', A_4A_1' and A_2A_5' are concurrent as well.

19. (**Simson's line**) If A, B, C are points on a circle, then the feet of perpendiculars from an arbitrary point D of that circle to the sides of ABC are collinear.

20. Let A, B, C, D be four points on a circle. Prove that the intersection of the Simsons line corresponding to A with respect to the triangle BCD and the Simsons line corresponding to B w.r.t. $\triangle ACD$ belongs to the line passing through C and the orthocenter of $\triangle ABD$.

21. Denote by $l(S;PQR)$ the Simsons line corresponding to the point S with respect to the triangle PQR. If the points A,B,C,D belong to a circle, prove that the lines $l(A;BCD)$, $l(B;CDA)$, $l(C,DAB)$, and $l(D,ABC)$ are concurrent.

22. (Taiwan 2002) Let A, B, and C be fixed points in the plane, and D the mobile point of the circumcircle of $\triangle ABC$. Let I_A denote the Simsons line of the point A with respect to $\triangle BCD$. Similarly we define I_B, I_C, and I_D. Find the locus of the points of intersection of the lines I_A, I_B, I_C, and I_D when D moves along the circle.

23. (BMO 2003) Given a triangle ABC, assume that $AB \neq AC$. Let D be the intersection of the tangent to the circumcircle of $\triangle ABC$ at A with the line BC. Let E and F be the points on the bisectors of the segments AB and AC respectively such that BE and CF are perpendicular to BC. Prove that the points D, E, and F lie on a line.

24. (**Pascal's Theorem**) If the hexagon $ABCDEF$ can be inscribed in a circle, prove that the points $AB \cap DE$, $BC \cap EF$, and $CD \cap FA$ are colinear.

25. (**Brokard's Theorem**) Let $ABCD$ be an inscribed quadrilateral. The lines AB and CD intersect at E, the lines AD and BC intersect in F, and the lines AC and BD intersect in G. Prove that O is the orthocenter of the triangle EFG.

26. (Iran 2005) Let ABC be an equilateral triangle such that $AB = AC$. Let P be the point on the extention of the side BC and let X and Y be the points on AB and AC such that

$$PX \parallel AC, \quad PY \parallel AB.$$

Let T be the midpoint of the arc BC. Prove that $PT \perp XY$.

27. Let $ABCD$ be an inscribed quadrilateral and let K, L, M, and N be the midpoints of AB, BC, CA, and DA respectively. Prove that the orthocenters of $\triangle AKN$, $\triangle BKL$, $\triangle CLM$, $\triangle DMN$ form a parallelogram.

6 Polygons Circumscribed Around Circle

Similarly as in the previous chapter, here we will assume that the unit circle is the one inscribed in the given polygon. Again we will make a use of theorem 2 and especially its third part. In the case of triangle we use also the formulas from the theorem 7. Notice that in this case we know both the incenter and circumcenter which was not the case in the previous section. Also, notice that the formulas from the theorem 7 are quite complicated, so it is highly recommended to have the circumcircle for as the unit circle whenever possible.

28. The circle with the center O is inscribed in the triangle ABC and it touches the sides AB, BC, CA in M, K, E respectively. Denote by P the intersection of MK and AC. Prove that $OP \perp BE$.

29. The circle with center O is inscribed in a quadrilateral $ABCD$ and touches the sides AB, BC, CD, and DA respectively in K, L, M, and N. The lines KL and MN intersect at S. Prove that $OS \perp BD$.

30. (BMO 2005) Let ABC be an acute-angled triangle which incircle touches the sides AB and AC in D and E respectively. Let X and Y be the intersection points of the bisectors of the angles $\angle ACB$ and $\angle ABC$ with the line DE. Let Z be the midpoint of BC. Prove that the triangle XYZ is isosceles if and only if $\angle A = 60^\circ$.

31. (**Newtons Theorem**) Given an circumscribed quadrilateral $ABCD$, let M and N be the midpoints of the diagonals AC and BD. If S is the incenter, prove that M, N, and S are colinear.

32. Let $ABCD$ be a quadrilateral whose incircle touches the sides AB, BC, CD, and DA at the points M, N, P, and Q. Prove that the lines AC, BD, MP, and NQ are concurrent.

33. (Iran 1995) The incircle of $\triangle ABC$ touches the sides BC, CA, and AB respectively in D, E, and F. X, Y, and Z are the midpoints of EF, FD, and DE respectively. Prove that the incenter of $\triangle ABC$ belongs to the line connecting the circumcenters of $\triangle XYZ$ and $\triangle ABC$.

34. Assume that the circle with center I touches the sides BC, CA, and AB of $\triangle ABC$ in the points D, E, F, respectively. Assume that the lines AI and EF intersect at K, the lines ED and KC at L, and the lines DF and KB at M. Prove that LM is parallel to BC.

35. (25. Tournament of Towns) Given a triangle ABC, denote by H its orthocenter, I the incenter, O its circumcenter, and K the point of tangency of BC and the incircle. If the lines IO and BC are parallel, prove that AO and HK are parallel.

36. (IMO 2000) Let AH_1, BH_2, and CH_3 be the altitudes of the acute-angled triangle ABC. The incircle of ABC touches the sides BC, CA, AB respectively in T_1, T_2, and T_3. Let l_1, l_2, and l_3 be the lines symmetric to H_2H_3, H_3H_1, H_1H_2 with respect to T_2T_3, T_3T_1, and T_1T_2 respectively. Prove that the lines l_1, l_2, l_3 determine a triagnle whose vertices belong to the incircle of ABC.

7 The Midpoint of Arc

We often encounter problems in which some point is defined to be the midpoint of an arc. One of the difficulties in using complex numbers is distinguishing the arcs of the cirle. Namely, if we define the midpoint of an arc to be the intersection of the bisector of the corresponding chord with the circle, we are getting two solutions. Such problems can be relatively easy solved using the first part of the theorem 8. Moreover the second part of the theorem 8 gives an alternative way for solving the problems with incircles and circumcircles. Notice that the coordinates of the important points are given with the equations that are much simpler than those in the previous section. However we have a problem when calculating the points d, e, f of tangency of the incircle with the sides (calculate them!), so in this case we use the methods of the previous section. In the case of the non-triangular polygon we also prefer the previous section.

37. (Kvant M769) Let L be the incenter of the triangle ABC and let the lines AL, BL, and CL intersect the circumcircle of $\triangle ABC$ at A_1, B_1, and C_1 respectively. Let R be the circumradius and r the inradius. Prove that:

$$\text{(a)}\ \frac{LA_1 \cdot LC_1}{LB} = R; \quad \text{(b)}\ \frac{LA \cdot LB}{LC_1} = 2r; \quad \text{(c)}\ \frac{S(ABC)}{S(A_1B_1C_1)} = \frac{2r}{R}.$$

38. (Kvant M860) Let O and R be respectively the center and radius of the circumcircle of the triangle ABC and let Z and r be respectively the incenter and inradius of $\triangle ABC$. Denote by K the centroid of the triangle formed by the points of tangency of the incircle and the sides. Prove that Z belongs to the segment OK and that $OZ : ZK = 3R/r$.

39. Let P be the intersection of the diagonals AC and BD of the convex quadrilateral $ABCD$ for which $AB = AC = BD$. Let O and I be the circumcenter and incenter of the triangle ABP. Prove that if $O \neq I$ then $OI \perp CD$.

40. Let I be the incenter of the triangle ABC for which $AB \neq AC$. Let O_1 be the point symmetric to the circumcenter of $\triangle ABC$ with respect to BC. Prove that the points A, I, O_1 are colinear if and only if $\angle A = 60^\circ$.

41. Given a triangle ABC, let A_1, B_1, and C_1 be the midpoints of BC, CA, and AB respecctively. Let P, Q, and R be the points of tangency of the incircle k with the sides BC, CA, and AB. Let P_1, Q_1, and R_1 be the midpoints of the arcs QR, RP, and PQ on which the points P, Q, and R divide the circle k, and let P_2, Q_2, and R_2 be the midpoints of arcs QPR, RQP, and PRQ respectively. Prove that the lines A_1P_1, B_1Q_1, and C_1R_1 are concurrent, as well as the lines A_1P_1, B_1Q_2, and C_1R_2.

8 Important Points. Quadrilaterals

In the last three sections the points that we've taken as initial, i.e. those with *known* coordinates have been "equally improtant" i.e. all of them had the same properties (they've been either the points of the same circle, or intersections of the tangents of the same circle, etc.). However, there are numerous problems where it is possible to distinguish one point from the others based on its influence to the other points. That point will be regarded as the origin. This is particularly useful in the case of quadrilaterals (that can't be inscribed or circumscribed around the circle) – in that case the intersection of the diagonals can be a good choice for the origin. We will make use of the formulas from the theorem 9.

42. The squares $ABB'B''$, $ACC'C''$, $BCXY$ are consctructed in the exterior of the triangle ABC. Let P be the center of the square $BCXY$. Prove that the lines CB'', BC'', AP intersect in a point.

43. Let O be the intersection of diagonals of the quadrilateral $ABCD$ and M, N the midpoints of the side AB and CD respectively. Prove that if $OM \perp CD$ and $ON \perp AB$ then the quadrilateral $ABCD$ is cyclic.

44. Let F be the point on the base AB of the trapezoid $ABCD$ such that $DF = CF$. Let E be the intersection of AC and BD and O_1 and O_2 the circumcenters of $\triangle ADF$ and $\triangle FBC$ respectively. Prove that $FE \perp O_1O_2$.

45. (IMO 2005) Let $ABCD$ be a convex quadrilateral whose sides BC and AD are of equal length but not parallel. Let E and F be interior points of the sides BC and AD respectively such that $BE = DF$. The lines AC and BD intersect at P, the lines BD and EF intersect at Q, and the lines EF and AC intersect at R. Consider all such triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P.

46. Assume that the diagonals of $ABCD$ intersect in O. Let T_1 and T_2 be the centroids of the triangles AOD and BOC, and H_1 and H_2 orthocenters of $\triangle AOB$ and $\triangle COD$. Prove that $T_1T_2 \perp H_1H_2$.

9 Non-unique Intersections and Viete's formulas

The point of intersection of two lines can be determined from the system of two equations each of which corresponds to the condition that a point correspond to a line. However this method can lead us into some difficulties. As we mentioned before standard methods can lead to non-unique points. For example, if we want to determine the intersection of two circles we will get a quadratic equations. That is not surprising at all since the two circles have, in general, two intersection points. Also, in many of the problems we don't need both of these points, just the direction of the line determined by them. Similarly, we may already know one of the points. In both cases it is more convenient to use Vieta's formulas and get the sums and products of these points. Thus we can avoid "taking the square root of a complex number" which is very suspicious operation by itself, and usually requires some knowledge of complex analysis.

Let us make a remark: If we need explicitly coordinates of one of the intersection points of two circles, and we don't know the other, the only way to solve this problem using complex numbers is to set the given point to be one of the initial points.

47. Suppose that the tangents to the circle Γ at A and B intersect at C. The circle Γ_1 which passes through C and touches AB at B intersects the circle Γ at the point M. Prove that the line AM bisects the segment BC.

48. (Republic Competition 2004, 3rd grade) Given a circle k with the diameter AB, let P be an arbitrary point of the circle different from A and B. The projections of the point P to AB is Q. The circle with the center P and radius PQ intersects k at C and D. Let E be the intersection of CD and PQ. Let F be the midpoint of AQ, and G the foot of perpendicular from F to CD. Prove that $EP = EQ = EG$ and that A, G, and P are colinear.

49. (China 1996) Let H be the orthocenter of the triangle ABC. The tangents from A to the circle with the diameter BC intersect the circle at the points P and Q. Prove that the points P, Q, and H are colinear.

50. Let P be the point on the extension of the diagonal AC of the rectangle $ABCD$ over the point C such that $\angle BPD = \angle CBP$. Determine the ratio $PB : PC$.

51. (IMO 2004) In the convex quadrilateral $ABCD$ the diagonal BD is not the bisector of any of the angles ABC and CDA. Let P be the point in the interior of $ABCD$ such that

$$\angle PBC = \angle DBA \text{ and } \angle PDC = \angle BDA.$$

Prove that the quadrilateral $ABCD$ is cyclic if and only if $AP = CP$.

10 Different Problems – Different Methods

In this section you will find the problems that are not closely related to some of the previous chapters, as well as the problems that are related to more than one of the chapters. The useful advice is to carefully think of possible initial points, the origin, and the unit circle. As you will see, the main problem with solving these problems is the time. Thus if you are in competition and you want to use complex numbers it is very important for you to estimate the time you will spend. Having this in mind, it is very important to learn complex numbers as early as possible.

You will see several problems that use theorems 3, 4, and 5.

52. Given four circles k_1, k_2, k_3, k_4, assume that $k_1 \cap k_2 = \{A_1, B_1\}$, $k_2 \cap k_3 = \{A_2, B_2\}$, $k_3 \cap k_4 = \{A_3, B_3\}$, $k_4 \cap k_1 = \{A_4, B_4\}$. If the points A_1, A_2, A_3, A_4 lie on a circle or on a line, prove that the points B_1, B_2, B_3, B_4 lie on a circle or on a line.

53. Suppose that $ABCD$ is a parallelogram. The similar and equally oliented triangles CD and CB are constructed outside this parallelogram. Prove that the triangle FAE is similar and equally oriented with the first two.

54. Three triangles KPQ, QLP, and PQM are constructed on the same side of the segment PQ in such a way that $\angle QPM = \angle PQL = \alpha$, $\angle PQM = \angle QPK = \beta$, and $\angle PQK = \angle QPL = \gamma$. If $\alpha < \beta < \gamma$ and $\alpha + \beta + \gamma = 180^\circ$, prove that the triangle KLM is similar to the first three.

55. *(Iran, 2005) Let n be a prime number and H_1 a convex n-gon. The polygons $H_2, \dots, H_n$ are defined recurrently: the vertices of the polygon H_{k+1} are obtained from the vertices of H_k by symmetry through k-th neighbour (in the positive direction). Prove that H_1 and H_n are similar.

56. Prove that the area of the triangles whose vertices are feet of perpendiculars from an arbitrary vertex of the cyclic pentagon to its edges doesn't depend on the choice of the vertex.

57. The points A_1, B_1, C_1 are chosen inside the triangle ABC to belong to the altitudes from A, B, C respectively. If

$$S(ABC_1) + S(BCA_1) + S(CAB_1) = S(ABC),$$

prove that the quadrilateral $A_1B_1C_1H$ is cyclic.

58. (IMO Shortlist 1997) The feet of perpendiculars from the vertices A, B, and C of the triangle ABC are D, E, end F respectively. The line through D parallel to EF intersects AC and AB respectively in Q and R. The line EF intersects BC in P. Prove that the circumcircle of the triangle PQR contains the midpoint of BC.

59. (BMO 2004) Let O be a point in the interior of the acute-angled triangle ABC. The circles through O whose centers are the midpoints of the edges of $\triangle ABC$ mutually intersect at K, L, and M, (different from O). Prove that O is the incenter of the triangle KLM if and only if O is the circumcenter of the triangle ABC.

60. Two circles k_1 and k_2 are given in the plane. Let A be their common point. Two mobile points, M_1 and M_2 move along the circles with the constant speeds. They pass through A always at the same time. Prove that there is a fixed point P that is always equidistant from the points M_1 and M_2.

61. (Yug TST 2004) Given the square $ABCD$, let γ be i circle with diameter AB. Let P be an arbitrary point on CD, and let M and N be intersections of the lines AP and BP with γ that are different from A and B. Let Q be the point of intersection of the lines DM and CN. Prove that $Q \in \gamma$ and $AQ : QB = DP : PC$.

62. (IMO Shortlist 1995) Given the triangle ABC, the circle passing through B and C intersect the sides AB and AC again in C' and B' respectively. Prove that the lines BB', CC', and HH' are concurrent, where H and H' orthocenters of the triangles ABC and $A'B'C'$ respectively.

63. (IMO Shortlist 1998) Let M and N be interior points of the triangle ABC such that $\angle MAB = \angle NAC$ and $\angle MBA = \angle NBC$. Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

64. (IMO Shortlist 1998) Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360^\circ$ and $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$. Prove that

$$BC \cdot AE \cdot FD = CA \cdot EF \cdot DB.$$

65. (IMO Shortlist 1998) Let ABC be a triangle such that $\angle A = 90^\circ$ and $\angle B < \angle C$. The tangent at A to its circumcircle ω intersect the line BC at D. Let E be the reflection of A with respect to BC, X the foot of the perpendicular from A to BE, and Y the midpoint of AX. If the line BY intersects ω in Z, prove that the line BD tangents the circumcircle of $\triangle ADZ$.

Hint: Use some inversion first...

66. (Rehearsal Competition in MG 1997, 3-4 grade) Given a triangle ABC, the points A_1, B_1 and C_1 are located on its edges BC, CA, and AB respectively. Suppose that $\triangle ABC \sim \triangle A_1B_1C_1$. If either the orthocenters or the incenters of $\triangle ABC$ and $\triangle A_1B_1C_1$ coincide prove that the triangle ABC is equilateral.

67. (**Ptolomy's inequality**) Prove that for every convex quadrilateral $ABCD$ the following inequality holds

$$AB \cdot CD + BC \cdot AD \geq AC \cdot BD.$$

68. (China 1998) Find the locus of all points D such that

$$DA \cdot DB \cdot AB + DB \cdot DC \cdot BC + DC \cdot DA \cdot CA = AB \cdot BC \cdot CA.$$

11 Disadvantages of the Complex Number Method

The bigest difficulties in the use of the method of complex numbers can be encountered when dealing with the intersection of the lines (as we can see from the fifth part of the theorem 2, although it dealt with the chords of the circle). Also, the difficulties may arrise when we have more than one circle in the problem. Hence you should avoid using the comples numbers in problems when there are lot of lines in general position without some special circle, or when there are more then two circles. Also, the things can get very complicated if we have only two circles in general position, and only in the rare cases you are advised to use complex numbers in such situations. The problems when some of the conditions is the equlity with sums of distances between non-colinear points can be very difficult and pretty-much unsolvable with this method.

Of course, these are only the obvious situations when you can't count on help of complex numbers. There are numerous innocent-looking problems where the calculation can give us increadible difficulties.

12 Hints and Solutions

Before the solutions, here are some remarks:

- In all the problems it is assumed that the lower-case letters denote complex numbers corresponding to the points denoted by capital letters (sometimes there is an exception when the unit circle is the incircle of the triangle and its center is denoted by o).
- Some abbreviations are used for addressing the theorems. For example T1.3 denotes the third part of the theorem 1.
- The solutions are quite useless if you don't try to solve the problem by yourself.
- Obvious derivations and algebraic manipulations are skipped. All expressions that are somehow "equally" related to both a and b are probably divisible by $a-b$ or $a+b$.
- To make the things simpler, many conjugations are skipped. However, these are very straightforward, since most of the numbers are on the unit circle and they satisfy $\overline{a} = \frac{1}{a}$.
- If you still doesn't believe in the power of complex numbers, you are more than welcome to try these problems with other methods– but don't hope to solve all of them. For example, try the problem 41. Sometimes, complex numbers can give you shorter solution even when comparing to the elementar solution.
- The author has tried to make these solutions available in relatively short time, hence some mistakes are possible. For all mistakes you've noticed and for other solutions (with complex numbers), please write to me to the above e-mail address.

1. Assume that the circumcircle of the triangle abc is the unit circle, i.e. $s=0$ and $|a|=|b|=|c|=1$. According to T6.3 we have $h=a+b+c$, and according to T6.1 we conclude that $h+q=2s=0$, i.e. $q=-a-b-c$. Using T6.2 we get $t_1=\dfrac{b+c+q}{3}=-\dfrac{a}{3}$ and similarly $t_2=-\dfrac{b}{3}$ and $t_3=-\dfrac{c}{3}$. We now have $|a-t_1|=\left|a+\dfrac{a}{3}\right|=\left|\dfrac{4a}{3}\right|=\dfrac{4}{3}$ and similarly $|b-t_2|=|c-t_3|=\dfrac{4}{3}$. The proof is complete. We have assumed that $R=1$, but this is no loss of generality.

2. For the unit circle we will take the circumcircle of the quadrilateral $abcd$. According to T6.3 we have $h_a=b+c+d$, $h_b=c+d+a$, $h_c=d+a+b$, and $h_d=a+b+c$. In order to prove that $abcd$ and $h_ah_bh_ch_d$ are congruent it is enough to establish $|x-y|=|h_x-h_y|$, for all $x,y\in\{a,b,c,d\}$. This is easy to verify.

3. Notice that the point h ca be obtained by the rotation of the point a around b for the angle $\dfrac{\pi}{2}$ in the positive direction. Since $e^{i\frac{\pi}{2}}=i$, using T1.4 we get $(a-b)i=a-h$, i.e. $h=(1-i)a+ib$. Similarly we get $d=(1-i)b+ic$ and $g=(1-i)c+ia$. Since $BCDE$ is a square, it is a parallelogram as well, hence the midpoints of ce and bd coincide, hence by T6.1 we have $d+b=e+c$, or $e=(1+i)b-ic$. Similarly $g=(1+i)c-ia$. The quadrilaterals $beph$ and $cgqd$ are parallelograms implying that $p+b=e+h$ and $c+q=g+d$, or

$$p=ia+b-ic,\quad q=-ia+ib+c.$$

In order to finish the proof it is enough to show that q ca be obtained by the rotation of p around a by an angle $\dfrac{\pi}{2}$, which is by T1.4 equivalent to

$$(p-a)i=p-b.$$

The last identity is easy to verify.

4. The points b_1, c_1, d_1, are obtained by rotation of b, c, d around c, d, and a for the angle $\dfrac{\pi}{3}$ in the positive direction. If we denote $e^{i\pi/3}=\varepsilon$ using T1.4 we get

$$(b-c)\varepsilon=b_1-c,\quad (c-d)\varepsilon=c_1-d,\quad (d-a)\varepsilon=d_1-a.$$

Since p is the midpoint of b_1c_1 T6.1 gives

$$p=\frac{b_1+c_1}{2}=\frac{\varepsilon b+c+(1-\varepsilon)d}{2}.$$

Similarly we get $q=\dfrac{\varepsilon c+d+(1-\varepsilon)a}{2}$. Using T6.1 again we get $r=\dfrac{a+b}{2}$. It is enough to prove that q can be obtained by the rotation of p around r for the angle $\dfrac{\pi}{3}$, in the positive direction. The last is (by T1.4) equivalent to

$$(p-r)\varepsilon=q-r,$$

which follows from

$$p-r=\frac{-a+(\varepsilon-1)b+c+(1-\varepsilon)}{2},\quad q-r=\frac{-\varepsilon a-b+\varepsilon c+d}{2},$$

and $\varepsilon^2-\varepsilon+1=0$ (since $0=\varepsilon^3+1=(\varepsilon+1)(\varepsilon^2-\varepsilon+1)$).

5. Let $\varepsilon=e^{i\frac{2\pi}{3}}$. According to T1.4 we have $p_{k+1}-a_{k+1}=(p_k-a_{k+1})\varepsilon$. Hence

$$\begin{aligned} p_{k+1} &= \varepsilon p_k+(1-\varepsilon)a_{k+1}=\varepsilon(\varepsilon p_{k-1}+(1-\varepsilon)a_k)+(1-\varepsilon)a_{k+1}=\dots \\ &= \varepsilon^{k+1}p_0+(1-\varepsilon)\sum_{i=1}^{k+1}\varepsilon^{k+1-i}a_i. \end{aligned}$$

Now we have $p_{1996}=p_0+665(1-\varepsilon)(\varepsilon^2a_1+\varepsilon a_2+a_3)$, since $\varepsilon^3=1$. That means $p_{1996}=p_0$ if and only if $\varepsilon^2a_1+\varepsilon a_2+a_3=0$. Using that $a_1=0$ we conclude $a_3=-\varepsilon a_2$, and it is clear that a_2 can be obtained by the rotation of a_3 around $0=a_1$ for the angle $\dfrac{\pi}{3}$ in the positive direction.

6. Since the point a is obtained by the rotation of b around o_1 for the angle $\dfrac{2\pi}{3}=\varepsilon$ in the positive direction, T1.4 implies $(o_1-b)\varepsilon=o_1-a$, i.e. $o_1=\dfrac{a-b\varepsilon}{1-\varepsilon}$. Analogously

$$o_2=\frac{b-c\varepsilon}{1-\varepsilon},\quad o_3=\frac{c-d\varepsilon}{1-\varepsilon},\quad o_4=\frac{d-a\varepsilon}{1-\varepsilon}.$$

Since $o_1o_3\perp o_2o_4$ is equivalent to $\dfrac{o_1-o_3}{\overline{o_1}-\overline{o_3}}=-\dfrac{o_2-o_4}{\overline{o_2}-\overline{o_4}}$, it is enouogh to prove that

$$\frac{a-c-(b-d)\varepsilon}{\overline{a-c}-\overline{(b-d)\varepsilon}}=-\frac{b-d-(c-a)\varepsilon}{\overline{b-d}-\overline{(c-a)\varepsilon}},$$

i.e. that $(a-c)\overline{b-d}-(b-d)\overline{b-d}\varepsilon+(a-c)\overline{a-c}\overline{\varepsilon}-(b-d)\overline{a-c}\varepsilon\overline{\varepsilon}=-\overline{a-c}(b-d)+(b-d)\overline{b-d}\overline{\varepsilon}-(a-c)\overline{a-c}\varepsilon+(a-c)\overline{b-d}\varepsilon\overline{\varepsilon}$. The last follows from $\overline{\varepsilon}=\dfrac{1}{\varepsilon}$ and $|a-c|^2=(a-c)\overline{a-c}=|b-d|^2=(b-d)\overline{b-d}$.

7. We can assume that $a_k=\varepsilon^k$ for $0\leq k\leq 12$, where $\varepsilon=e^{i\frac{2\pi}{15}}$. By rotation of the points a_1, a_2, and a_4 around $a_0=1$ for the angles ω^6, ω^5, and ω^3 (here $\omega=e^{i\pi/15}$), we get the points a_1', a_2', and a_4', such that takve da su a_0,a_7,a_1',a_2',a_4' kolinearne. Sada je dovoljno dokazati da je

$$\frac{1}{a_1'-1}=\frac{1}{a_2'-1}+\frac{1}{a_4'-1}+\frac{1}{a_7-1}.$$

From T1.4 we have $a_1'-a_0=(a_1-a_0)\omega^6, a_2'-a_0=(a_2-a_0)\omega^5$ and $a_4'-a_0=(a_4-a_0)\omega^3$, as well as $\varepsilon=\omega^2$ and $\omega^{30}=1$. We get

$$\frac{1}{\omega^6(\omega^2-1)}=\frac{1}{\omega^5(\omega^4-1)}+\frac{1}{\omega^3(\omega^8-1)}-\frac{\omega^{14}}{\omega^{16}-1}.$$

Taking the common denominator and cancelling with ω^2-1 we see that it is enough to prove that

$$\omega^8+\omega^6+\omega^4+\omega^2+1=\omega(\omega^{12}+\omega^8+\omega^4+1)+\omega^3(\omega^8+1)-\omega^{20}.$$

Since $\omega^{15}=-1=-\omega^{30}$, we have that $\omega^{15-k}=-\omega^{30-k}$. The required statement follows from $0=\omega^{28}+\omega^{26}+\omega^{24}+\omega^{22}+\omega^{20}+\omega^{18}+\omega^{16}+\omega^{14}+\omega^{12}+\omega^{10}+\omega^8+\omega^6+\omega^4+\omega^2+1=\frac{\omega^{30}-1}{\omega^2-1}=0$.

8. [Obtained from Uroš Rajković] Take the complex plane in which the center of the polygon is the origin and let $z=e^{i\frac{\pi}{k}}$. Now the coordinate of A_k in the complex plane is z^{2k}. Let p ($|p|=1$) be the coordinate of P. Denote the left-hand side of the equality by S. We need to prove that $S=\dbinom{2m}{m}\cdot n$. We have that

$$S=\sum_{k=0}^{n-1}PA_k^{2m}=\sum_{k=0}^{n-1}\left|z^{2k}-p\right|^{2m}$$

Notice that the arguments of the complex numbers $(z^{2k}-p)\cdot z^{-k}$ (where $k\in\{0,\,1,\,2,\,\dots,n\}$) are equal to the argument of the complex number $(1-p)$, hence

$$\frac{(z^{2k}-p)\cdot z^{-k}}{1-p}$$

is a positive real number. Since $|z^{-k}| = 1$ we get:

$$S = \sum_{k=0}^{n-1} |z^{2k} - p|^{2m} = |1-p|^{2m} \cdot \sum_{k=0}^{n-1} \left(\frac{z^{2k} - p}{1-p} \right)^{2m} = |1-p|^{2m} \cdot \frac{\sum_{k=0}^{n-1} (z^{2k} - p)^{2m}}{(1-p)^{2m}}.$$

Since S is a positive real number we have:

$$S = \left| \sum_{k=0}^{n-1} (z^{2k} - p)^{2m} \right|.$$

Now from the binomial formula we have:

$$S = \left| \sum_{k=0}^{n-1} \left[\sum_{i=0}^{2m} \binom{2m}{i} \cdot z^{2ki} \cdot (-p)^{2m-i} \right] \cdot z^{-2mk} \right|.$$

After some algebra we get:

$$S = \left| \sum_{k=0}^{n-1} \sum_{i=0}^{2m} \binom{2m}{i} \cdot z^{2k(i-m)} \cdot (-p)^{2m-i} \right|,$$

or, equivalently

$$S = \left| \sum_{i=0}^{2m} \binom{2m}{i} \cdot (-p)^{2m-i} \cdot \sum_{k=0}^{n-1} z^{2k(i-m)} \right|.$$

Since for $i \neq m$ we have:

$$\sum_{k=0}^{n-1} z^{2k(i-m)} = \frac{z^{2n(i-m)} - 1}{z^{2(i-m)} - 1},$$

for $z^{2n(i-m)} - 1 = 0$ and $z^{2(i-m)} - 1 \neq 0$, we have

$$\sum_{k=0}^{n-1} z^{2k(i-m)} = 0.$$

For $i = m$ we have:

$$\sum_{k=0}^{n-1} z^{2k(i-m)} = \sum_{k=0}^{n-1} 1 = n.$$

From this we conclude:

$$S = \left| \binom{2m}{m} \cdot (-p)^m \cdot n \right| = \binom{2m}{m} \cdot n \cdot |(-p)^m|.$$

Using $|p| = 1$ we get

$$S = \binom{2m}{m} \cdot n$$

and that is what we wanted to prove.

9. Choose the circumcircle of the triangle abc to be the unit circle. Then $o = 0$ and $\overline{a} = \frac{1}{a}$. The first of the given relations can be written as

$$1 = \frac{|a-m||b-n|}{|a-n||b-m|} \Rightarrow 1 = \frac{|a-m|^2 |b-n|^2}{|a-n|^2 |b-m|^2} = \frac{(a-m)(\overline{a} - \overline{m})(a-n)(\overline{a} - \overline{n})}{(a-n)(\overline{a} - \overline{n})(b-m)(\overline{b} - \overline{m})}$$

After some simple algebra we get $(a-m)(\overline{a}-\overline{m})(b-n)(\overline{b}-\overline{n}) = (1-\frac{m}{a}-a\overline{m}+m\overline{m})(1-\frac{n}{b}-b\overline{n}+n\overline{n}) = 1-\frac{m}{a}-a\overline{m}+m\overline{m}-\frac{n}{b}+\frac{mn}{ab}+\frac{a\overline{m}n}{b}-\frac{m\overline{m}n}{b}-b\overline{n}+\frac{bm\overline{n}}{a}+ab\overline{m}\overline{n}-bm\overline{m}\overline{n}+n\overline{n}-\frac{mn\overline{n}}{a}-a\overline{m}n\overline{n}+m\overline{m}n\overline{n}$. The value of the expression $(a-n)(\overline{a}-\overline{n})(b-m)(\overline{b}-\overline{m})$ we can get from the prevoius one replacing every a with b and vice versa. The initial equality now becomes:

$$\begin{aligned} & 1-\frac{m}{a}-a\overline{m}+m\overline{m}-\frac{n}{b}+\frac{mn}{ab}+\frac{a\overline{m}n}{b}-\frac{m\overline{m}n}{b}-b\overline{n}+ \\ & \frac{bm\overline{n}}{a}+ab\overline{m}\overline{n}-bm\overline{m}\overline{n}+n\overline{n}-\frac{mn\overline{n}}{a}-a\overline{m}n\overline{n}+m\overline{m}n\overline{n} \\ = \ & 1-\frac{m}{b}-b\overline{m}+m\overline{m}-\frac{n}{a}+\frac{mn}{ab}+\frac{b\overline{m}n}{a}-\frac{m\overline{m}n}{a}-a\overline{n}+\frac{am\overline{n}}{b}+ \\ & ab\overline{m}\overline{n}-am\overline{m}\overline{n}+n\overline{n}-\frac{mn\overline{n}}{b}-b\overline{m}n\overline{n}+m\overline{m}n\overline{n}. \end{aligned}$$

Subtracting and taking $a-b$ out gives

$$\frac{m}{ab}-\overline{m}-\frac{n}{ab}+\frac{(a+b)\overline{m}n}{ab}-\frac{m\overline{m}n}{ab}+\overline{n}-\frac{(a+b)m\overline{n}}{ab}+m\overline{m}\overline{n}+\frac{mn\overline{n}}{ab}-\overline{m}n\overline{n}=0.$$

Since $AM/CM = AN/CM$ holds as well we can get the expression analogous to the above when every b is exchanged with c. Subtracting this expression from the previous and taking $b-c$ out we get

$$-\frac{m}{abc}+\frac{n}{abc}-\frac{\overline{m}n}{bc}+\frac{m\overline{m}n}{abc}+\frac{m\overline{n}}{bc}-\frac{mn\overline{n}}{abc}=0.$$

Writing the same expression with ac instead of bc (this can be obtained from the initial conditions because of the symmetry), subtracting, and simplifying yields $m\overline{n}-n\overline{m}=0$. Now we have $\frac{m-o}{\overline{m}-\overline{o}} = \frac{n-o}{\overline{n}-\overline{o}}$, and by T1.2 the points m,n,o are colinear.

10. [Obtained from Uroš Rajković] First we will prove that for the points p, a, and b of the unit circle the distance from p to the line ab is equal to:

$$\frac{1}{2}|(a-p)(b-p)|.$$

Denote by q the foot of perpendicular from p to ab and use T2.4 to get:

$$q=\frac{1}{2}\Big(p+a+b-\frac{ab}{p}\Big).$$

Now the required distance is equal to:

$$|q-p|=\frac{1}{2}\left|-p+a+b-\frac{ab}{p}\right|.$$

Since $|p|=1$ we can multiply the expression on the right by $-p$ which gives us:

$$\left|\frac{1}{2}(p^2-(a+b)p+ab)\right|.$$

Now it is easy to see that the required distance is indeed equal to:

$$\frac{1}{2}|(a-p)(b-p)|.$$

If we denote $z=e^{i\frac{2\pi}{2n}}$, the coordinate of A_k is z^{2k}. Now we have:

$$2\cdot h_k=|(z^{2k}-p)(z^{2k-2}-p)|.$$

The vector $(z^{2k}-p)\cdot z^{-k}$ is colinear with $1-p$, nece

$$\frac{(z^{2k}-p)\cdot z^{-k}}{1-p}$$

is a positive real number. Hence for $k\in\{1,2,\cdots,n-1\}$ it holds:

$$h_k=\frac{(z^{2k}-p)\cdot(z^{2k-2}-p)\cdot z^{-(2k-1)}}{2\cdot(1-p)^2}\cdot|1-p|^2,$$

since $|z|=1$. We also have:

$$h_n=\frac{(1-p)\cdot(z^{2n-2}-p)\cdot z^{-(n-1)}}{2\cdot(1-p)^2}\cdot|1-p|^2.$$

We need to prove that:

$$\sum_{k=1}^{n-1}\frac{1}{\frac{(z^{2k}-p)\cdot(z^{2k-2}-p)\cdot z^{-(2k-1)}}{2\cdot(1-p)^2}\cdot|1-p|^2}=\frac{1}{\frac{(1-p)\cdot(z^{2n-2}-p)\cdot z^{-(n-1)}}{2\cdot(1-p)^2}\cdot|1-p|^2}.$$

After cancelling and multiplying by z we get:

$$\sum_{k=1}^{n-1}\frac{z^{2k}}{(z^{2k}-p)\cdot(z^{2k-2}-p)}=\frac{-1}{(1-p)\cdot(z^{2n-2}-p)},$$

since $z^n=-1$. Denote by S the left-hand side of the equality. We have:

$$S-\frac{1}{z^2}S=\sum_{k=1}^{n-1}\frac{(z^{2k}-p)-(z^{2k-2}-p)}{(z^{2k}-p)\cdot(z^{2k-2}-p)}.$$

This implies:

$$(1-\frac{1}{z^2})S=\sum_{k=1}^{n-1}\left(\frac{1}{z^{2k-2}-p}-\frac{1}{z^{2k}-p}\right).$$

After simplifying we get:

$$(1-\frac{1}{z^2})S=\frac{1}{1-p}-\frac{1}{z^{2n-2}-p}=\frac{(z^{2n-2}-p)-(1-p)}{(1-p)\cdot(z^{2n-2}-p)}.$$

Since $z^{2n-2}=\frac{1}{z^2}$ (from $z^n=1$) we get:

$$S=\frac{-1}{(1-p)\cdot(z^{2n-2}-p)},$$

q.e.d.

11. Assume that the unit circle is the circumcircle of the quadrilateral $abcd$. Since ac is its diameter we have $c=-a$. Furthermore by T2.5 we have that

$$m=\frac{ab(c+d)-cd(a+b)}{ab-cd}=\frac{2bd+ad-ab}{d+b}.$$

According to T2.3 we have that $n=\frac{2bd}{b+d}$, hence $m-n=\frac{a(d-b)}{b+d}$ and $\overline{m}-\overline{n}=\frac{b-d}{a(b+d)}$. Now we have

$$\frac{m-n}{\overline{m}-\overline{n}}=-\frac{a-c}{\overline{a}-\overline{c}}=a^2,$$

hence according to T1.3 $mn\perp ac$, q.e.d.

12. Assume that the unit circle is the circumcircle of the triangle abc. Using T6.3 we have $h=a+b+c$, and using T2.4 we have $e=\frac{1}{2}\Big(a+b+c-\frac{ac}{b}\Big)$. Since $paqb$ is a parallelogram the midpoints of pq and ab coincide, and according to T6.1 $q=a+b-p$ and analogously $r=a+c-p$. Since the points x,a,q are colinear, we have (using T1.2)

$$\frac{x-a}{\overline{x}-\overline{a}}=\frac{a-q}{\overline{a}-\overline{q}}=\frac{p-b}{\overline{p}-\overline{b}}=-pb,$$

or, equivalently $\overline{x}=\frac{pb+a^2-ax}{abp}$. Since the points h,r,x are colinear as well, using the same theorem we get

$$\frac{x-h}{\overline{x}-\overline{h}}=\frac{h-r}{\overline{h}-\overline{r}}=\frac{b+p}{\overline{b}+\overline{p}}=bp,$$

i.e.

$$\overline{x}=\frac{x-a-b-c+p+\frac{bp}{a}+\frac{bp}{c}}{bp}.$$

Equating the expressions obtained for $\overline{x}$ we get

$$x=\frac{1}{2}\Big(2a+b+c-p-\frac{bp}{c}\Big).$$

By T1.1 it is sufficient to prove that

$$\frac{e-x}{\overline{e}-\overline{x}}=\frac{a-p}{\overline{a}-\overline{p}}=-ap.$$

The last follows from

$$e-x=\frac{1}{2}\Big(p+\frac{bp}{c}-a-\frac{ac}{b}\Big)=\frac{bcp+b^2p-abc-ac^2}{2bc}=\frac{(b+c)(bp-ac)}{2bc},$$

by conjugation.

13. We will assume that the circumcircle of the quadrilateral $abcd$ is the unit circle. Using T2.4 and T6.1 we get

$$p=a+b-\frac{ab}{c},\quad q=a+d+\frac{ad}{c}\quad (1).$$

Let H be the orthocenter of the triangle ABD. By T6.3 we have $h=a+b+d$, hence according to T1.2 it is enough to prove that

$$\frac{p-h}{\overline{p}-\overline{h}}=\frac{q-h}{\overline{q}-\overline{h}}.\quad (2)$$

Chaning for p from (1) we get

$$\frac{p-h}{\overline{p}-\overline{h}}=\frac{a+b-\frac{ab}{c}-a-b-d}{\frac{1}{a}+\frac{1}{b}-\frac{c}{ab}-\frac{1}{a}-\frac{1}{b}-\frac{1}{d}}=\frac{abd}{c},$$

and since this expression is symmetric with respect to b and d, (2) is clearly satisfied.

14. Assume that the unit circle is the circumcircle of the triangle abc and assume that a',b',c' are feet of perpendiculars from a,b,c respectively. From T2.4 we have

$$a'=\frac{1}{2}\Big(a+b+c-\frac{bc}{a}\Big),\quad b'=\frac{1}{2}\Big(a+b+c-\frac{ca}{b}\Big),\quad c'=\frac{1}{2}\Big(a+b+c-\frac{ab}{c}\Big).$$

Since a',b',c' are midpoints of ad,be,cf respectively according to T6.1 we have

$$d=b+c-\frac{bc}{a},\quad e=a+c-\frac{ac}{b},\quad f=a+b-\frac{ab}{c}.$$

By T1.2 the colinearity of the points d,e,f is equivalent to

$$\frac{d-e}{\overline{d}-\overline{e}}=\frac{f-e}{\overline{f}-\overline{e}}.$$

Since $d-e=b-a+\frac{ac}{b}-\frac{bc}{a}=(b-a)\frac{ab-c(a+b)}{ab}$ and similarly $f-e=(b-c)\frac{bc-a(b+c)}{bc}$, by conjugation and some algebra we get

$$\begin{aligned}0&=(a^2b+a^2c-abc)(c-a-b)-(c^2a+c^2b-abc)(a-b-c)\\&=(c-a)(abc-a^2b-ab^2-a^2c-ac^2-b^2c-bc^2).\quad(1)\end{aligned}$$

Now we want to get the necessary and sufficient condition for $|h|=2$ (the radius of the circle is 1). After the squaring we get

$$\begin{aligned}4&=|h|^2=h\overline{h}=(a+b+c)\Big(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\Big)\\&=\frac{a^2b+ab^2+a^2c+ac^2+b^2c+bc^2+3abc}{abc}.\quad(2)\end{aligned}$$

Now (1) is equivalent to (2), which finishes the proof.

15. Assume that the unit circle is the circumcircle of the triangle abc. Let a',b',c' be the midpoints of bc,ca,ab. Since $aa_1\perp ao$ and since a_1,b',c' are colinear, using T1.3 and T1.2, we get

$$\frac{a-a_1}{\overline{a}-\overline{a_1}}=-\frac{a-o}{\overline{a}-\overline{o}}=-a^2,\quad\frac{b'-c'}{\overline{b'}-\overline{c'}}=\frac{b'-a_1}{\overline{b'}-\overline{a_1}}.$$

From the first equality we have $\overline{a_1}=\frac{2a-a_1}{a^2}$, and since from T6.1 $b'=\frac{a+c}{2}$ and $c'=\frac{a+b}{2}$ we also have $\overline{a_1}=\frac{ab+bc+ca-aa_1}{2abc}$. By equating the above expressions we get $a_1=\frac{a^2(a+b+c)-3abc}{a^2-2bc}$. Similarly $b_1=\frac{b^2(a+b+c)-3abc}{2(b^2-ac)}$ and $c_1=\frac{c^2(a+b+c)-3abc}{2(c^2-2ab)}$. Now we have

$$a_1-b_1=\frac{a^2(a+b+c)-3abc}{2(a^2-bc)}-\frac{b^2(a+b+c)-3abc}{2(b^2-ac)}=-\frac{c(a-b)^3(a+b+c)}{2(a^2-bc)(b^2-ac)},$$

and it is easy to verify the condition for $a_1b_1 \perp ho$, which is according to T1.3:

$$\frac{a_1-b_1}{\overline{a_1}-\overline{b_1}} = -\frac{h-o}{\overline{h}-\overline{o}} = -\frac{(a+b+c)abc}{ab+bc+ca}.$$

Similarly $a_1c_1 \perp ho$, implying that the points a_1, a_2, and a_3 are colinear.

16. Assume that the unit circle is the circumcircle of the triangle abc. By T2.4 we have that $b_1 = \frac{1}{2}\Big(a+b+c-\frac{ac}{b}\Big)$ and $c_1 = \frac{1}{2}\Big(a+b+c-\frac{ab}{c}\Big)$, according to T6.1 $m = \frac{b+c}{2}$, and according to T6.3 $h = a+b+c$. Now we will determine the point d. Since d belongs to the chord bc according to T2.2 $\overline{d} = \frac{b+c-d}{bc}$. Furthermore, since the points b_1, c_1, and d are colinear, according to T1.2 we have

$$\frac{d-b_1}{\overline{d}-\overline{b_1}} = \frac{b_1-c_1}{\overline{b_1}-\overline{c_1}} = \frac{a\Big(\frac{b}{c}-\frac{c}{b}\Big)}{\frac{1}{a}\Big(\frac{c}{b}-\frac{b}{c}\Big)} = -a^2.$$

Now we have that $\overline{d} = \frac{a^2\overline{b_1}+b_1-d}{a^2}$, hence

$$d = \frac{a^2b+a^2c+ab^2+ac^2-b^2c-bc^2-2abc}{2(a^2-bc)}.$$

In order to prove that $dh \perp am$ (see T1.3) it is enough to prove that $\frac{d-h}{\overline{d}-\overline{h}} = -\frac{m-a}{\overline{m}-\overline{a}}$. This however follows from

$$\begin{aligned} d-h &= \frac{b^2c+bc^2+ab^2+ac^2-a^2b-a^2c-2a^3}{2(a^2-bc)} \\ &= \frac{(b+c-2a)(ab+bc+ca+a^2)}{2(a^2-bc)} \end{aligned}$$

and $m-a = \frac{b+c-2a}{2}$ by conjugation.

17. Assume that the unit circle is the circumcircle of the triangle abc. By T2.4 we have that $f = \frac{1}{2}\Big(a+b+c-\frac{ab}{c}\Big)$. Since a,c,p are colinear and ac is a chord of the unit circle, according to T2.2 we have $\overline{p} = \frac{a+c-p}{ac}$. Since $fo \perp pf$ using T1.3 we coclude

$$\frac{f-o}{\overline{f}-\overline{o}} = -\frac{p-f}{\overline{p}-\overline{f}}.$$

From the last two relations we have

$$p = f\frac{2ac\overline{f}-(a+c)}{ac\overline{f}-f} = \frac{\Big(a+b+c-\frac{ab}{c}\Big)c^2}{b^2+c^2}.$$

Let $\angle phf = \varphi$, then

$$\frac{f-h}{\overline{f}-\overline{h}} = e^{i2\varphi}\frac{p-h}{\overline{p}-\overline{h}}.$$

Since $p-h = -b\frac{ab+bc+ca+c^2}{b^2+c^2}$, and by conjugation

$$\overline{p}-\overline{h} = -\frac{c(ab+bc+ca+b^2)}{ab(b^2+c^2)},$$

$f-h=\dfrac{ab+bc+ca+c^2}{2c}$, $\overline{f}-\overline{h}=\dfrac{ab+bc+ca+c^2}{2abc}$, we see that $e^{i2\varphi}=\dfrac{c}{b}$. On the other hand we have $\dfrac{c-a}{\overline{c}-\overline{a}}=e^{i2\alpha}\dfrac{b-a}{\overline{b}-\overline{a}}$, and using T1.2 $e^{i2\alpha}=\dfrac{c}{b}$. We have proved that $\alpha=\pi+\varphi$ or $\alpha=\varphi$, and since the first is impossible, the proof is complete.

18. First we will prove the following useful lemma.

Lemma 1. *If a, b, c, a', b', c' are the points of the unit circle, then the lines aa', bb', cc' concurrent or colinear if and only if*

$$(a-b')(b-c')(c-a')=(a-c')(b-a')(c-b').$$

Proof. Let x be the intersection of aa' and bb', and let y be the intersection of the lines aa' and cc'. Using T2.5 we have

$$x=\frac{aa'(b+b')-bb'(a+a')}{aa'-bb'},\quad y=\frac{aa'(c+c')-cc'(a+a')}{aa'-cc'}.$$

Here we assumed that these points exist (i.e. that none of $aa'\parallel bb'$ and $aa'\parallel cc'$ holds). It is obvious that the lines aa', bb', cc' are concurrent if and only if $x=y$, i.e. if and only if

$$(aa'(b+b')-bb'(a+a'))(aa'-cc')=(aa'(c+c')-cc'(a+a'))(aa'-bb').$$

After simplifying we get $aa'b+aa'b'-abb'-a'b'b-bcc'-b'cc'=aa'c+aa'c'-bc'c-bb'c'-acc'-a'cc'$, and since this is equivalent to $(a-b')(b-c')(c-a')=(a-c')(b-a')(c-b')$, the lemma is proven. □

Now assume that the circumcircle of the hexagon is the unit circle. Using T1.1 we get

$$\frac{a_2-a_4}{\overline{a_2}-\overline{a_4}}=\frac{a_0-a_0'}{\overline{a_0}-\overline{a_0'}},\quad \frac{a_4-a_0}{\overline{a_4}-\overline{a_0}}=\frac{a_2-a_2'}{\overline{a_2}-\overline{a_2'}},\quad \frac{a_2-a_0}{\overline{a_2}-\overline{a_0}}=\frac{a_4-a_4'}{\overline{a_4}-\overline{a_4'}},$$

hence $a_0'=\dfrac{a_2a_4}{a_0}, a_2'=\dfrac{a_0a_4}{a_2}$ i $a_4'=\dfrac{a_0a_2}{a_4}$. Similarly, using T2.5 we get

$$a_3'=\frac{a_0'a_3(a_2+a_3)-a_2a_3(a_0'+a_3)}{a_0'a_3-a_2a_4}=\frac{a_4(a_3-a_2)+a_3(a_2-a_0)}{a_3-a_0}.$$

Analogously,

$$a_5'=\frac{a_0(a_5-a_4)+a_5(a_4-a_2)}{a_5-a_2},\quad a_1'=\frac{a_2(a_1-a_0)+a_1(a_0-a_4)}{a_1-a_4}.$$

Assume that the points a_3'', a_1'', a_5'' are the other intersection points of the unit circle with the lines a_0a_3', a_4a_1', a_2a_5' respectively. According to T1.2

$$\frac{a_3'-a_0}{\overline{a_3'}-\overline{a_0}}=\frac{a_3''-a_0}{\overline{a_3''}-\overline{a_0}}=-a_3''a_0,$$

and since $a_0-a_3'=\dfrac{a_3(2a_0-a_2-a_4)+a_2a_4-a_0^2}{a_3-a_0}$, we have

$$a_3''-a_4=\frac{(a_0-a_2)^2(a_3-a_4)}{a_0a_2(a_3-a_0)(\overline{a_0}-\overline{a_3'})},\quad a_3''-a_2=\frac{(a_0-a_4)^2(a_3-a_2)}{a_0a_4(a_3-a_0)(\overline{a_0}-\overline{a_3'})}.$$

Analogously we get

$$a_1''-a_0=a_3''-a_4=\frac{(a_2-a_4)^2(a_1-a_0)}{a_2a_4(a_1-a_4)(\overline{a_4}-\overline{a_1'})},$$

$$a_1''-a_2=a_3''-a_4=\frac{(a_4-a_0)^2(a_1-a_2)}{a_0a_4(a_1-a_4)(\overline{a_4}-\overline{a_1'})},$$
$$a_5''-a_0=a_3''-a_4=\frac{(a_2-a_4)^2(a_5-a_0)}{a_2a_4(a_5-a_0)(\overline{a_2}-\overline{a_5'})},$$
$$a_5''-a_4=a_3''-a_4=\frac{(a_0-a_2)^2(a_5-a_4)}{a_0a_2(a_5-a_4)(\overline{a_2}-\overline{a_5'})}.$$

Using the lemma and the concurrence of the lines a_0a_3, a_1a_4, and a_2a_5 (i.e. $(a_0-a_1)(a_2-a_3)(a_4-a_5)=(a_0-a_5)(a_2-a_1)(a_4-a_3)$) we get the concurrence of the lines a_0a_3'', a_4a_1'', and a_2a_5'', i.e. $(a_0-a_1'')(a_2-a_3'')(a_4-a_5'')=(a_0-a_5'')(a_2-a_1'')(a_4-a_3'')$, since they, obviously, intersect.

19. [Obtained from Uroš Rajković] Assume that the unit circle is the circumcircle of the triangle abc. If A_1, B_1, and C_1 denote the feet of the perpendiculars, we have from T2.4:

$$a_1 = \frac{1}{2}\left(b+c+m-\frac{bc}{m}\right),$$
$$b_1 = \frac{1}{2}\left(a+c+m-\frac{ac}{m}\right), \text{ and}$$
$$c_1 = \frac{1}{2}\left(a+b+m-\frac{ab}{m}\right).$$

We further get:

$$\frac{a_1-c_1}{b_1-c_1}=\frac{c-a+\dfrac{ab-bc}{m}}{c-b+\dfrac{ab-ac}{m}}=\frac{(c-a)(m-b)}{(c-b)(m-a)}=\frac{\overline{a}_1-\overline{c}_1}{\overline{b}_1-\overline{c}_1},$$

and, according to T1.2, the points A_1, B_1, and C_1 are colinear.

20. The quadrilateral $ABCD$ is cyclic, and we assume that it's circumcircle is the unti circle. Let a_1, a_2, and a_3 denote the feet of the perpendiculars from a to bc, cd, and db respectively. Denote by b_1, b_2, and b_3 the feet of the perpendiculars from b to ac, cd, and da respectively. According to T2.4 we have that

$$a_1=\frac{1}{2}\left(a+b+c-\frac{bc}{a}\right),\ a_2=\frac{1}{2}\left(a+b+d-\frac{bd}{a}\right),\ a_3=\frac{1}{2}\left(a+c+d-\frac{cd}{a}\right)$$
$$b_1=\frac{1}{2}\left(b+a+c-\frac{ac}{b}\right),\ b_2=\frac{1}{2}\left(b+c+d-\frac{cd}{b}\right),\ b_3=\frac{1}{2}\left(b+d+a-\frac{da}{b}\right)$$

The point x can be obtained from the condition for colinearity. First from the colinearity of x, a_1, a_2 and T1.2 we have that

$$\frac{x-a_1}{\overline{x}-\overline{a_1}}=\frac{a_1-a_2}{\overline{a_1}-\overline{a_2}}=\frac{\frac{1}{2}\left(c-d+\frac{bd}{a}-\frac{bc}{a}\right)}{\frac{1}{2}\left(\frac{1}{c}-\frac{1}{d}+\frac{a}{bd}-\frac{a}{bc}\right)}=\frac{bcd}{a},$$

and after simplifying

$$\overline{x}=\frac{x-\frac{1}{2}\left(a+b+c+d-\frac{abc+acd+abd+bcd}{a^2}\right)}{bcd}a.$$

Similarly from the colinearity of the points x, b_1, and b_2 we get

$$\overline{x}=\frac{x-\frac{1}{2}\left(a+b+c+d-\frac{abc+acd+abd+bcd}{b^2}\right)}{acd}b,$$

and from this we conclude

$$x=\frac{1}{2}\Big(a+b+c+d\Big).$$

Let $h=a+c+d$ (by T6) be the orthocenter of the triangle acd. In order to finish the proof, according to T1.2 it is enough to show that

$$\frac{x-c}{\overline{x}-\overline{c}}=\frac{h-c}{\overline{h}-\overline{c}}=\frac{a+b+d-c}{\overline{a}+\overline{b}+\overline{d}-\overline{c}}.$$

On the other hand $x-c=\frac{1}{2}\Big(a+b+d-c\Big)$, from which the equality is obvious.

21. Using the last problem we have that the intersection of the lines $l(a;bcd)$ and $l(b;cda)$ is the point $x=\frac{1}{2}\Big(a+b+c+d\Big)$, which is a symmetric expression, hence this point is the intersection of every two of the given lines.

22. Using the last two problems we get the locus of points is the set of all the points of the form $x=\frac{1}{2}\Big(a+b+c+d\Big)$, when d moves along the circle. That is in fact the circle with the radius $\frac{1}{2}$ and center $\frac{a+b+c}{2}$, which is the midpoint of the segment connecting the center of the given circle with the orthocenter of the triangle abc.

23. Assume that the unit circle is the circumcircle of the triangle abc. From T1.3 and the condition $ad\perp ao$ we have that

$$\frac{d-a}{\overline{d}-\overline{a}}=-\frac{a-o}{\overline{a}-\overline{o}}=-a^2,$$

and after simplifying $\overline{d}=\frac{2a-d}{a^2}$. Since the points b,c,d are colinear and bc is the chord of the unit circle, according to T2.2 $\overline{d}=\frac{b+c-d}{bc}$, and solving the given system we get $d=\frac{a^2(b+c)-2abc}{a^2-bc}$. Since e belongs to the perpendicular bisector of ab we have $oe\perp ab$. According to T1.3 and $\frac{e-o}{\overline{e}-\overline{o}}=-\frac{a-b}{\overline{a}-\overline{b}}=ab$, i.e. $\overline{e}=\frac{e}{ab}$. From $be\perp bc$, using T1.3 again we get $\frac{b-e}{\overline{b}-\overline{e}}=-\frac{b-c}{\overline{b}-\overline{c}}=bc$, or equivalently $\overline{e}=\frac{c-b+e}{bc}=\frac{e}{ab}$. Hence $e=\frac{a(c-b)}{c-a}$. Similarly we have $f=\frac{a(b-c)}{b-a}$. Using T1.2 we see that it is enough to prove that $\frac{d-f}{\overline{d}-\overline{f}}=\frac{f-e}{\overline{f}-\overline{e}}$. Notice that

$$\begin{aligned}d-f &= \frac{a^2(b+c)-2abc}{a^2-bc}-\frac{a(b-c)}{b-a}=\frac{a^2b^2+3a^2bc-ab^2c-2a^3b-abc^2}{(a^2-bc)(b-a)}\\ &= \frac{ab(a-c)(b+c-2a)}{(a^2-bc)(b-a)},\end{aligned}$$

and similarly $d-e=\frac{ac(a-b)(b+c-2a)}{(a^2-bc)(c-a)}$. After conjugation we see that the required condition is easy to verify.

24. [Obtained from Uroš Rajković] Assume that the unit circle is the incircle of the hexagon $ABCDEF$. After conjugating and using T2.5 we get:

$$\overline{m}=\frac{a+b-(d+e)}{ab-de},\ \overline{n}=\frac{b+c-(e+f)}{bc-ef},\ \overline{p}=\frac{c+d-(f+a)}{cd-fa},$$

hence:

$$\overline{m}-\overline{n}=\frac{(b-e)(bc-cd+de-ef+fa-ab)}{(ab-de)(bc-ef)},$$

and analogously:

$$\overline{n}-\overline{p}=\frac{(c-f)(cd-de+ef-fa+ab-bc)}{(bc-ef)(cd-fa)}.$$

From here we get:

$$\frac{\overline{m}-\overline{n}}{\overline{n}-\overline{p}}=\frac{(b-e)(cd-fa)}{(f-c)(ab-de)}.$$

Since the numbers $\overline{a}$, $\overline{b}$, $\overline{c}$, $\overline{d}$, $\overline{e}$, and $\overline{f}$ are equal to $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, $\frac{1}{d}$, $\frac{1}{e}$, and $\frac{1}{f}$, respectively, we see that it is easy to verify that the complex number on the left-hand side of the last equality equal to its complex conjugate, hence it is real. Now according to T1.2 the points M, N, and P are colinear, q.e.d.

25. Assume that the quadrilateral $abcd$ is inscribed in the unit circle. Using T2.5 we get

$$\begin{aligned} e &= \frac{ab(c+d)-cd(a+b)}{ab-cd}, \\ f &= \frac{ad(b+c)-bc(a+d)}{ad-bc} mboxand \\ g &= \frac{ac(b+d)-bd(a+c)}{ac-bd}. \quad (1) \end{aligned}$$

In order to prove that $o=0$ is the orthocenter of the triangle efg, it is enough to prove that $of \perp eg$ and $og \perp ef$. Because of the symmetry it is enough to prove one of these two relateions. Hence, by T1.3 it is enough to prove that

$$\frac{f-o}{\overline{f}-\overline{o}}=\frac{e-g}{\overline{e}-\overline{g}} \quad (2).$$

From (1) we have that

$$\frac{f-o}{\overline{f}-\overline{o}}=\frac{\dfrac{ad(b+c)-bc(a+d)}{ad-bc}}{\dfrac{(b+c)-(a+d)}{bc-ad}}=\frac{ad(b+c)-bc(a+d)}{a+d-(b+c)},(3)$$

or equivalently

$$\begin{aligned} e-g &= \frac{(a-d)(ab^2d-ac^2d)+(b-c)(bcd^2-a^2bc)}{(ab-cd)(ac-bd)} \\ &= \frac{(a-d)(b-c)((b+c)ad-(a+d)bc)}{(ab-cd)(ac-bd)} \quad (4) \end{aligned}$$

and by conjugation

$$\overline{e}-\overline{g}=\frac{(a-d)(b-c)(b+c-(a+d))}{(ab-cd)(ac-bd)} \quad (5).$$

Comparing the expressions (3),(4), and (5) we derive the statement.

26. Assume that the unit circle is the circumcircle of the triangle abc and assume that $a=1$. Then $c=\overline{b}$ and $t=-1$. Since p belongs to the chord bc, using T2.2 we get that $\overline{p}=b+\frac{1}{b}-p$. Since x belongs to the chord ab, in the similar way we get $\overline{x}=\frac{1+b-x}{b}$. Since $px \parallel ac$ by T1.1 we have

$$\frac{p-x}{\overline{p}-\overline{x}}=\frac{a-c}{\overline{a}-\overline{c}}=-\frac{1}{b},$$

i.e. $\overline{x} = pb + \overline{p} - xb$. From this we get $x = \dfrac{b(p+1)}{b+1}$. Similarly we derive $y = \dfrac{p+1}{b+1}$. According to T1.3 it remains to prove that $\dfrac{x-y}{\overline{x}-\overline{y}} = -\dfrac{p-t}{\overline{p}-\overline{t}} = -\dfrac{p+1}{\overline{p}+1}$. This follows from $x - y = \dfrac{(p+1)(b-1)}{b+1}$ and by conjugation

$$\overline{x} - \overline{y} = \frac{(\overline{p}+1)\left(\frac{1}{b}-1\right)}{\frac{1}{b}+1} = -\frac{(\overline{p}+1)(b-1)}{b+1}.$$

27. Assume that the unit circle is the circumcircle of the quadrilateral $abcd$. Using T6.1 we have $k = \dfrac{a+b}{2}$, $l = \dfrac{b+c}{2}$, $m = \dfrac{c+a}{2}$ and $n = \dfrac{d+a}{2}$. We want to determine the coordinate of the orthocenter of the triangle akn. Let h_1 be that point and denote by h_2, h_3, and h_4 the orthocenters of bkl, clm, and dmn respectively. Then $kh_1 \perp an$ and $nh_1 \perp ak$. By T1.3 we get

$$\frac{k-h_1}{\overline{k}-\overline{h_1}} = -\frac{a-n}{\overline{a}-\overline{n}} \quad \text{and} \quad \frac{n-h_1}{\overline{n}-\overline{h_1}} = -\frac{a-k}{\overline{a}-\overline{k}}. \qquad (1)$$

Since

$$\frac{a-n}{\overline{a}-\overline{n}} = \frac{a-d}{\overline{a}-\overline{d}} = -ad,$$

we have that

$$\overline{h_1} = \frac{\overline{k}ad - k + h_1}{ad}.$$

Similarly from the second of the equations in (1) we get

$$\overline{h_1} = \frac{\overline{n}ab - n + h_1}{ab}.$$

Solving this system gives us that

$$h_1 = \frac{2a+b+d}{2}.$$

Symmetricaly

$$h_2 = \frac{2b+c+a}{2},\ h_3 = \frac{2c+d+b}{2},\ h_4 = \frac{2d+a+c}{2},$$

and since $h_1 + h_3 = h_2 + h_4$ using T6.1 the midpoints of the segments h_1h_3 and h_2h_4 coincide hence the quadrilateral $h_1h_2h_3h_4$ is a parallelogram.

28. Assume that the unit circle is the circumcircle of the triangle abc. By T2.3 we have that $a = \dfrac{2em}{e+m}$ i $b = \dfrac{2mk}{m+k}$. Let's find the point p. Since the points m, k, and p are colinear and mk is the chord of the unit circle, by T2.2 we have that $\overline{p} = \dfrac{m+k-p}{mk}$. Furthermore the points p, e, and c are colinear. However, in this problem it is more convenient to notice that $pe \perp oe$ and now using T1.3 we have

$$\frac{e-p}{\overline{e}-\overline{p}} = -\frac{e-o}{\overline{e}-\overline{o}} = -e^2$$

and after simplifying $\overline{p} = \dfrac{2e-p}{e^2}$. Equating the two expressions for $\overline{p}$ we get

$$p = e\frac{(m+k)e-2mk}{e^2-mk}.$$

In order to finish the proof using T1.3 it is enough to prove that $\frac{p-o}{\overline{p}-\overline{o}} = -\frac{e-b}{\overline{e}-\overline{b}}$. This will follow from

$$e-b = \frac{e(m+k)-2mk}{m+k},$$

and after conjugating $\overline{e}-\overline{b} = \frac{m+k-2e}{(m+k)e}$ and $\overline{p} = \frac{m+k-2e}{mk-e^2}$.

29. Assume that the circle inscribed in $abcd$ is the unit one. From T2.3 we have that

$$a = \frac{2nk}{n+k}, \quad b = \frac{2kl}{k+l}, \quad c = \frac{2lm}{l+m}, \quad d = \frac{2mn}{m+n}. \quad (1)$$

Using T2.5 we get

$$s = \frac{kl(m+n)-mn(k+l)}{kl-mn}. \quad (2)$$

According to T1.1 it is enough to verify that

$$\frac{s-o}{\overline{s}-\overline{o}} = \frac{b-d}{\overline{b}-\overline{d}}.$$

From (1) we have that

$$b-d = 2\frac{kl(m+n)-mn(k+l)}{(k+l)(m+n)}, \quad (3)$$

and after conjugating

$$\overline{b}-\overline{d} = \frac{m+n-(k+l)}{(k+l)(m+n)}. \quad (4)$$

From (2) we have that

$$\frac{s}{\overline{s}} = \frac{kl(m+n)-mn(k+l)}{kl-mn}, \quad (5)$$

and comparing the expressions (3),(4), and (5) we finish the proof.

30. [Obtained from Uroš Rajković] Let P be the point of tangency of the incircle with the line BC. Assume that the incircle is the unit circle. By T2.3 the coordinates of A, B, and C are respectively

$$a = \frac{2qr}{q+r}, \quad b = \frac{2pr}{p+r} \text{ i } c = \frac{2pq}{p+q}.$$

Furthermore, using T6.1 we get $x = \frac{1}{2}(b+c) = \frac{pr}{p+r} + \frac{pq}{p+q}$, $y = \alpha b = \alpha\frac{2pr}{p+r}$, and $z = \beta c = \beta\frac{2pq}{p+q}$ $(\alpha, \beta \in R)$. The values of α and β are easy to compute from the conditions $y \in rq$ and $z \in rq$:

$$\alpha = \frac{(p+r)(q+r)}{2(p+q)r} \text{ i } \beta = \frac{(p+q)(r+q)}{2(p+r)q}.$$

From here we get the coordinates of y and z using p, q, and r:

$$y = \frac{p(q+r)}{(p+q)} \text{ and } z = \frac{p(r+q)}{(p+r)}.$$

We have to prove that:

$$\angle RAQ = 60^\circ \iff XYZ \text{ is equilateral.}$$

The first condition is equivalent to $\angle QOR = 60^\circ$ i.e. with

$$r = q \cdot e^{i2\pi/3}.$$

The second condition is equivalent to $(z-x) = (y-x)\cdot e^{i\pi/3}$. Notice that:

$$y - x = \frac{p(q+r)}{(p+q)} - \left(\frac{pr}{p+r} + \frac{pq}{p+q}\right) = \frac{pr(r-q)}{(p+q)(p+r)} \text{ and}$$

$$z - x = \frac{p(p+q)}{(p+r)} - \left(\frac{pr}{p+r} + \frac{pq}{p+q}\right) = \frac{pq(q-r)}{(p+q)(p+r)}.$$

Now the second condition is equivalent to:

$$\frac{pq(q-r)}{(p+q)(p+r)} = \frac{pr(r-q)}{(p+q)(p+r)}\, e^{i\pi/3},$$

i.e. with $q = -r\, e^{i\pi/3}$. It remains to prove the equivalence:

$$r = q\, e^{i2\pi/3} \Longleftrightarrow q = -r\, e^{i\pi/3},$$

which obviously holds.

31. According to T1.1 it is enough to prove that

$$\frac{m-o}{\overline{m}-\overline{o}} = \frac{n-o}{\overline{n}-\overline{o}}.$$

If p,q,r,s are the points of tangency of the incircle with the sides ab, bc, cd, da respectively using T2.3 we get

$$m = \frac{a+c}{2} = \frac{ps}{p+s} + \frac{qr}{q+r} = \frac{pqs+prs+pqr+qrs}{(p+s)(q+r)},$$

and after conjugating $\overline{m} = \dfrac{p+q+r+s}{(p+s)(q+r)}$ and

$$\frac{m}{\overline{m}} = \frac{pqr+ps+prs+qrs}{p+q+r+s}.$$

Since the last expression is symmetric in p,q,r,s we conclude that $\dfrac{m}{\overline{m}} = \dfrac{n}{\overline{n}}$, as required.

32. Assume that the incircle of the quadrilateral $abcd$ is the unit circle. We will prove that the intersection of the lines mp and nq belongs to bd. Then we can conlude by symmetry that the point also belongs to ac, which will imply that the lines mp, nq, ac, and bd are concurrent. Using T2.3 we have that

$$b = \frac{2mn}{m+n}, \quad d = \frac{2pq}{p+q}.$$

If x is the intersection point of mp and nq, using T2.5 we get

$$x = \frac{mp(n+q) - nq(m+p)}{mp - nq}.$$

We have to prove that the points x, b, d are colinear, which is according to T1.2 equivalent to saying that

$$\frac{b-d}{\overline{b}-\overline{d}} = \frac{b-x}{\overline{b}-\overline{x}}.$$

This follows from $b-d=\frac{2mn}{m+n}-\frac{2pq}{p+q}=2\frac{mn(p+q)-pq(m+n)}{(m+n)(p+q)}$ and

$$\begin{aligned} b-x &= \frac{2mn}{m+n}-\frac{mp(n+q)-nq(m+p)}{mp-nq} \\ &= \frac{m^2np-mn^2q-m^2pq+n^2pq+m^2nq-mn^2p}{(mp-nq)(m+n)} \\ &= \frac{(m-n)(mn(p+q)-pq(m+n))}{(m+n)(mp-nq)}, \end{aligned}$$

by conjugation.

33. Assume that the unit circle is the incumcircle of the triangle abc. Using T7.3 we have that the circumecenter has the coordinate

$$o=\frac{2def(d+e+f)}{(d+e)(e+f)(f+d)}.$$

Let's calculate the coordinate of the circumcenter o_1 of the triangle xyz. First, according to T6.1 we have that $x=\frac{e+f}{2}$, $y=\frac{d+f}{2}$ and $z=\frac{d+e}{2}$. Moreover by T1.3 we have that $\frac{o_1-\frac{x+y}{2}}{\overline{o_1}-\overline{\frac{x+y}{2}}}=-\frac{x-y}{\overline{x}-\overline{y}}=\frac{(e-d)/2}{(\overline{e}-\overline{d})/2}=-ed$, and simplifying

$$\overline{o_1}=\frac{-\frac{f}{2}+\frac{ed}{2f}+o_1}{ed},$$

and similarly $\overline{o_1}=\frac{-\frac{d}{2}+\frac{ef}{2d}+o_1}{ef}$. By equating we get $o_1=\frac{e+f+d}{2}$. Now by T1.2 it is enough to prove that $\frac{o_1-i}{\overline{o_1}-\overline{i}}=\frac{o-i}{\overline{o}-\overline{i}}$, which can be easily obtained by conjugation of the previous expressions for o and o_1.

34. Assume that the incircle of the triangle abc is the unit circle. Using T7.1 we get $b=\frac{2fd}{f+d}$ and $c=\frac{2ed}{e+d}$. From some elemetary geometry we conclude that k is the midpoint of segment ef hence by T6.1 we have $k=\frac{e+f}{2}$. Let's calculate the coordinate of the point m. Since m belongs to the chord fd by T2.2 we have $\overline{m}=\frac{f+d-m}{fd}$. Similarly we have that the points b,m,k are colinear and by T1.2 we get $\frac{k-m}{\overline{k}-\overline{m}}=\frac{b-k}{\overline{b}-\overline{k}}$, i.e. $\overline{m}=m\frac{\overline{b}-\overline{k}}{b-k}+\frac{\overline{k}b-k\overline{b}}{b-k}$. Now equating the expressions for $\overline{m}$ one gets

$$m=\frac{(f+d)(b-k)+(k\overline{b}-\overline{k}b)fd}{(\overline{b}-\overline{k})fd+b-k}.$$

Since $b-k=\frac{3fd-de-f^2-ef}{2(f+d)}$ and $k\overline{b}-\overline{k}b=\frac{(e+f)(e-d)fd}{e(f+d)}$ we get

$$m=\frac{4ef^2d+efd^2-e^2d^2-e^2f^2-2f^2d^2-f^3e}{6efd-e^2d-ed^2-ef^2-e^2f-d^2f-df^2}$$

and symmetrically

$$n=\frac{4e^2fd+efd^2-f^2d^2-e^2f^2-2e^2d^2-e^3f}{6efd-e^2d-ed^2-ef^2-e^2f-d^2f-df^2}.$$

By T1.3 it is enough to prove that $\frac{m-n}{\overline{m}-\overline{n}}=-\frac{i-d}{\overline{i}-\overline{d}}=-d^2$. This however follows from

$$m-n=\frac{(e-f)(4efd-ed^2-fd^2-fe^2-f^2e)}{6efd-e^2d-ed^2-ef^2-e^2f-d^2f-df^2},$$

by conjugation.

35. Assume that the unit circle is the inrcumcircle of the triangle abc. Assume that k, l, and m are the points of tangency of the incircle with the sides bc, ca, and ab, respectively. By T7 we have that

$$o=\frac{2klm(k+l+m)}{(k+l)(l+m)(m+k)},\quad h=\frac{2(k^2l^2+l^2m^2+m^2k^2+klm(k+l+m))}{(k+l)(l+m)(m+k)}.$$

Since the segments io and bc are parallel we have that $io\perp ik$, which is by T1.3 equivalent to $\frac{o-i}{\overline{o}-\overline{i}}=-\frac{k-i}{\overline{k}-\overline{i}}=-k^2$. After conjugating the last expression for o becomes

$$klm(k+l+m)+k^2(kl+lm+mk)=0.\quad (*)$$

Let's prove that under this condition we have $ao\parallel hk$. According to T1.1 it is enough to prove that $\frac{a-o}{\overline{a}-\overline{o}}=\frac{h-k}{\overline{h}-\overline{k}}$. According to T7.1 we have that $a=\frac{2ml}{m+l}$, and

$$a-o=\frac{2ml}{m+l}-\frac{2klm(k+l+m)}{(k+l)(l+m)(m+k)}=\frac{2m^2l^2}{(k+l)(l+m)(m+k)}.$$

Now we get that it is enough to prove that

$$\frac{h-k}{\overline{h}-\overline{k}}=\frac{l^2m^2}{k^2}.$$

Notice that

$$\begin{aligned}
h-k &= \frac{2(k^2l^2+l^2m^2+m^2k^2+klm(k+l+m))}{(k+l)(l+m)(m+k)}-k\\
&= \frac{k^2l^2+k^2m^2+2l^2m^2+k^2lm+kl^2m+klm^2-k^2l-k^3m-k^2lm}{(k+l)(l+m)(m+k)}\\
&= \frac{klm(k+l+m)-k^2(k+l+m)+k^2l^2+2l^2m^2+m^2l^2}{(k+l)(l+m)(m+k)}\\
&= \Big(\text{according to (*)}\Big)=\frac{(kl+lm+mk)^2+l^2m^2}{(k+l)(l+m)(m+k)}\\
&= \Big(\text{according to (*)}\Big)=\frac{(kl+lm+mk)^2((k+l+m)^2+k^2)}{(k+l+m)^2(k+l)(l+m)(m+k)}.
\end{aligned}$$

After conjugating the last expression for $h-k$ we get

$$\overline{h}-\overline{k}=\frac{(k+l+m)^2+k^2}{(k+l)(l+m)(m+k)},$$

and using the last expression for $h-k$ we get

$$\frac{h-k}{\overline{h}-\overline{k}}=\frac{(kl+lm+mk)^2}{(k+l+m)^2}=\Big(\text{by (*)}\Big)=\frac{l^2m^2}{k^2},$$

which completes the proof.

36. Assume that the incircle of the triangle abc is the unit circle. Then using T7.1 we have $c=\dfrac{2t_1t_2}{t_1+t_2}$. Our goal is to first determine the point h_3. From $h_3t_3\perp it_3$ by T1.3 we have

$$\frac{h_3-t_3}{\overline{h_3}-\overline{t_3}}=-\frac{t_3-i}{\overline{t_3}-\overline{i}}=-t_3^2,$$

i.e. $\overline{h_3}=\dfrac{2t_3-h_3}{t_3^2}$. Furthermore from $ch_3 \parallel it_3$ and T1.1 we have $\dfrac{h_3-c}{\overline{h_3}-\overline{c}}=\dfrac{t_3-i}{\overline{t_3}-\overline{i}}=t_3^2$. Writing the similar expression for $\overline{h_3}$ gives

$$h_3=\frac{1}{2}\Big(2t_3+c-\overline{c}t_3^2\Big)=t_3+\frac{t_1t_2-t_3^2}{t_1+t_2}.$$

Similarly we obtain $h_2=t_2+\dfrac{t_1t_3-t_2^2}{t_1+t_3}$. In order to determine the line symmetric to h_2h_3 with repsect to t_2t_3 it is enough to determine the points symmetric to h_2 and h_3 with respect to t_2t_3. Assume that p_2 and p_3 are these two points and let h_2' and h_3' be the feet of perpendiculars from h_2 and h_3 to the line t_2t_3 respectively. According to T2.4 we have $h_2'=\dfrac{1}{2}\Big(t_2+t_3-t_2t_3\overline{h_3}\Big)$ hence by T6.1

$$p_2=2h_2'-h_2=\frac{t_1(t_2^2+t_3^2)}{t_2(t_1+t_3)}$$

and symmetrically $p_3=\dfrac{t_1(t_2^2+t_3^2)}{t_3(t_1+t_2)}$. Furthermore

$$p_2-p_3=\frac{t_1^2(t_2^2+t_3^2)(t_3-t_2)}{t_1t_3(t_1+t_2)(t_1+t_3)},$$

and if the point x belongs to p_2p_3 by T1.2 the following must be satisfied:

$$\frac{x-p_2}{\overline{x}-\overline{p_2}}=\frac{p_2-p_3}{\overline{p_2}-\overline{p_3}}=-t_1^2.$$

Specifically if x belongs to the unit circle we also have $\overline{x}=\dfrac{1}{x}$, hence we get the quadratic equation

$$t_2t_3x^2-t_1(t_2^2+t_3^2)x+t_1^2t_2t_3=0.$$

Its solutions are $x_1=\dfrac{t_1t_2}{t_3}$ and $x_2=\dfrac{t_1t_3}{t_2}$ and these are the intersection points of the line p_2p_3 with the unit circle. Similarly we get $y_1=\dfrac{t_1t_2}{t_3}$, $y_2=\dfrac{t_2t_3}{t_1}$, and $z_1=\dfrac{t_3t_1}{t_2}$,$z_2=\dfrac{t_2t_3}{t_1}$, which finishes the proof.

37. Assume that the circumcircle of the triangle abc is the unit circle. Let u,v,w be the complex numbers described in T8. Using this theorem we get that $l=-(uv+vw+wu)$. By elementary geometry we know that the intersection of the line al and the circumcircle of the triangle abc is the midpoint of the arc bc which doesn't contain the point a. That means $a_1=-vw$ and similarly $b_1=-uw$ and $c_1=-uv$.

(a) The statement follows from the equality
$$1=\frac{|l-a_1|\cdot|l-c_1|}{|l-b|}=\frac{|u(v+w)|\cdot|w(u+v)|}{|uv+uw+vw+v^2|}=\frac{|v+w|\cdot|u+v|}{|(u+v)(v+w)|}=1.$$

(b) If x is the point of the tangency of the incircle with the side bc then x is the foot of the perpendicular from the point l l to the side bc and T2.4 implies $x=\frac{1}{2}\left(b+c+l-bc\overline{l}\right)$ and consequently $r=|l-x|=\frac{1}{2}\left|\frac{(u+v)(v+w)(w+u)}{u}\right|=\frac{1}{2}|(u+v)(v+w)(w+u)|$. Now the required equality follows from
$$\begin{aligned}\frac{|l-a|\cdot|l-b|}{|l-c_1|}&=\frac{|(u+v)(u+w)|\cdot|(u+v)(v+w)|}{|w(u+v)|}\\&=|(u+v)(v+w)(w+u)|.\end{aligned}$$

(c) By T5 we have that
$$S(ABC)=\frac{i}{4}\begin{vmatrix}u^2&1/u^2&1\\v^2&1/v^2&1\\w^2&1/w^2&1\end{vmatrix}\ \text{i}\ S(A_1B_1C_1)=\frac{i}{4uvw}\begin{vmatrix}vw&u&1\\uw&v&1\\uv&w&1\end{vmatrix},$$
hence
$$\begin{aligned}\frac{S(ABC)}{S(A_1B_1C_1)}&=\frac{u^4w^2+w^4v^2+v^4u^2-v^4w^2-u^4v^2-w^4u^2}{uvw(v^2w+uw^2+u^2v-uv^2-u^2w-vw^2)}\\&=\frac{(u^2-v^2)(uw+vw-uv-w^2)(uw+vw+uv+w^2)}{uvw(u-v)(uv+w^2-uw-vw)}\\&=-\frac{(u+w)(vw+uw+uv+w^2)}{uvw}\\&=-\frac{(u+v)(v+w)(w+u)}{uvw}.\end{aligned}$$
Here we consider the oriented surface areas, and substracting the modulus from the last expression gives us the desired equality.

38. First solution. Assume that the circumcircle of the triangle abc is the unit circle and u,v,w are the complex numbers described in T8. Let d,e,f be the points of tangency of the incircle with the sides bc,ca,ab respectively. By T2.4 we have that $f=\frac{1}{2}\left(a+b+z-ab\overline{z}\right)=\frac{1}{2}\Big(u^2+v^2+w^2-uv-vw-wu+\frac{uv(u+v)}{2w}\Big)$. By symmetry we get the expressions for e and f and by T6.1 we get
$$\begin{aligned}k&=\frac{1}{3}\left(u^2+v^2+w^2-uv-vw-wu+\frac{uv(u+v)}{2w}+\frac{vw(v+w)}{2u}-\frac{wu(w+u)}{2v}\right)=\\&=\frac{(uv+vw+wu)(u^2v+uv^2+uw^2+u^2w+v^2w+vw^2-4uvw)}{6uvw}.\end{aligned}$$
Now it is easy to verify $\frac{z-o}{\overline{z}-\overline{o}}=\frac{k-o}{\overline{k}-\overline{o}}$, which is by T1.2 the condition for colinearity of the points z,k,o. Similarly we also have
$$\begin{aligned}\frac{|o-z|}{|z-k|}&=\frac{|uv+vw+wu|}{\left|\frac{(uv+vw+wu)(u^2v+uv^2+uw^2+u^2w+v^2w+vw^2+2uvw)}{6uvw}\right|}\\&=\frac{6}{|(u+v)(v+w)(w+u)|}=\frac{6R}{2r}=\frac{3R}{r},\end{aligned}$$

which completes the proof.

Second solution. Assume that the incircle of the triangle abc is the unit circle and let d,e,f denote its points of tangency with the sides bc,ca,ab respectively. According to T7.3 we have that $o=\frac{2def(d+e+f)}{(d+e)(e+f)(f+d)}$ and according to T6.1 $k=\frac{d+e+f}{3}$. Now it is easy to verify that $\frac{o-z}{\overline{o}-\overline{z}}=\frac{k-z}{\overline{k}-\overline{z}}$ which is by T1.2 enough to establish the colinearity of the points o,z,k. We also have that

$$\frac{|o-z|}{|z-k|}=\frac{\left|\frac{d+e+f}{(d+e)(e+f)(f+d)}\right|}{\left|\frac{d+e+f}{3}\right|}=\frac{3}{|(d+e)(e+f)(f+d)|}=\frac{3R}{r}.$$

39. Assume that the circumcircle of the triangle abc is the unit circle and let u,v,w be the complex numbers described in T8 (here $p=w^2$). According to this theorem we have $i=-uv-vw-wu$. Since $|a-c|=|a-b|$ by T1.4 it holds

$$c-a=e^{i\angle cab}(b-a).$$

By the same theorem we have

$$\frac{-vw-u^2}{\overline{-vw}-\overline{u^2}}=e^{i2\frac{\angle pab}{2}}\frac{v^2-u^2}{\overline{v^2}-\overline{u^2}},$$

hence $e^{i\angle pab}=-\frac{w}{v}$. Now we have

$$c=\frac{u^2w+u^2v-v^2w}{v},$$

and symmetrically $d=\frac{v^2w+v^2u-u^2w}{u}$. By T1.3 it is enough to prove that

$$\frac{c-d}{\overline{c}-\overline{d}}=-\frac{o-i}{\overline{o}-\overline{i}}=-\frac{uv+vw+wu}{u+v+w}uvw.$$

This follows from $c-d=\frac{(u^2-v^2)(uv+vw+wu)}{uv}$ by conjugation.

40. Assume that the circumcircle of the triangle abc is the unit circle. By T8 there are numbers u,v,w such that $a=u^2,b=v^2,c=w^2$ and the incenter is $i=-(uv+vw+wu)$. If o' denotes the foot of the perpendicular from o to bc then by T2.4 we have $o'=\frac{1}{2}\Big(b+c\Big)$, and by T6.1 $o_1=2o'=b+c=v^2+w^2$. By T1.2 the points a,i,o_1 are colinear if and only if

$$\frac{o_1-a}{\overline{o_1}-\overline{a}}=\frac{a-i}{\overline{a}-\overline{i}}.$$

Since

$$\frac{o_1-a}{\overline{o_1}-\overline{a}}=\frac{o_1-a}{\overline{o_1}-\overline{a}}=\frac{v^2+w^2-u^2}{u^2(v^2+w^2)-v^2w^2}u^2v^2w^2 \text{ and}$$

$$\frac{a-i}{\overline{a}-\overline{i}}=\frac{u(u+v+w)+vw}{vw+uw+uv+u^2}u^2vw=u^2vw,$$

we get

$$v^3w+vw^3-u^2vw-(u^2v^2+u^2w^2-v^2w^2)=(vw-u^2)(v^2+w^2+vw)=0.$$

This means that either $vw = u^2$ or $v^2 + w^2 + vw = 0$. If $vw = u^2$ then by T6.1 the points u^2 and $-vw$ belong to the same radius hence abc is isosceles contrary to the assumption. This means that $v^2+w^2+vw=0$. We now want to prove that the triangle with the vertices $o, -vw, w^2$ is equilateral. It is enough to prove that $1 = |w^2+vw| = |v+w|$ which is equivalent to $1 = (v+w)(\overline{v}+\overline{w}) = \frac{(v+w)^2}{vw}$ and this to $v^2+w^2+vw=0$. Since $\angle boc = 120^\circ$ we have $\alpha = 60^\circ$.

41. Assume that the incumcircle of the triangle abc is the unit circle. According to T8 there are complex numbers u, v, w such that $p = u^2, q = v^2, r = w^2$ and $p_1 = -vw, q_1 = -wu, r_1 = -uv$. Then $p_2 = vw, q_2 = wu, r_2 = uv$. By T7.1 we gave

$$a = \frac{2v^2w^2}{v^2+w^2},\ b = \frac{2w^2u^2}{w^2+u^2}\ \text{i}\ c = \frac{2u^2w^2}{u^2+w^2},$$

hence by T6.1

$$a_1 = \frac{w^2u^2}{w^2+u^2} + \frac{u^2v^2}{u^2+v^2},\ b_1 = \frac{u^2v^2}{u^2+v^2} + \frac{v^2w^2}{v^2+w^2},\ c_2 = \frac{v^2w^2}{v^2+w^2} + \frac{w^2u^2}{w^2+u^2}.$$

If the point n is the intersection of the lines a_1p_1 and b_1q_1 then the triplets of points (n, a_1, p_1) and (n, b_1, q_1) are colinear and using T1.2 we get

$$\frac{n-a_1}{\overline{n}-\overline{a_1}} = \frac{a_1-p_1}{\overline{a_1}-\overline{p_1}},\quad \frac{n-b_1}{\overline{n}-\overline{b_1}} = \frac{b_1-q_1}{\overline{b_1}-\overline{q_1}}.$$

Solving this system gives us

$$\begin{aligned} n \;=\; & \frac{u^4v^4+v^4w^4+w^4u^4}{(u^2+v^2)(v^2+w^2)(w^2+u^2)} + \\ & \frac{uvw(u^3v^2+u^2v^3+u^3w^2+u^2w^3+v^3w^2+v^2w^3)}{(u^2+v^2)(v^2+w^2)(w^2+u^2)} + \\ & \frac{3u^2v^2w^2(u^2+v^2+w^2)}{(u^2+v^2)(v^2+w^2)(w^2+u^2)} + \\ & \frac{2u^2v^2w^2(uv+vw+wu)}{(u^2+v^2)(v^2+w^2)(w^2+u^2)}. \end{aligned}$$

Since the above expression is symmetric this point belongs to c_1r_1. The second part of the problem can be solved similarly.

42. Assume that a is the origin. According to T1.4 we have $c'' - a = e^{i\pi/2}(c-a)$, i.e. $c'' = ic$. Similarly we get $b'' = -ib$. Using the same theorem we obtain $x - c = e^{i\pi/2}(b-c)$, i.e. $x = (1-i)c + ib$ hence by T6.1 $p = \frac{1+i}{2}b + \frac{1-i}{2}c$. Denote by q the intersection of the lines bc and ap. Then the points a, p, q are colinear as well as the points b, c'', q. Using T1.2 we get

$$\frac{a-p}{\overline{a}-\overline{p}} = \frac{a-q}{\overline{a}-\overline{q}},\quad \frac{b-c''}{\overline{b}-\overline{c''}} = \frac{q-b}{\overline{q}-\overline{b}}.$$

From the first equation we conclude that $\overline{q} = q\frac{(1-i)\overline{b}+(1+i)\overline{c}}{(1+i)b+(1-i)c}$, and from the second we get the formula $\overline{q} = \frac{q(\overline{b}+i\overline{c}) - i(\overline{b}c+b\overline{c})}{b-ic}$. These two imply

$$q = \frac{i(\overline{b}c+b\overline{c})((1+i)b+(1-i)c)}{2(ib\overline{b}-2b\overline{c}+2\overline{b}c+2ic\overline{c})} = \frac{(\overline{b}c+b\overline{c})((1+i)b+(1-i)c)}{(b-ic)(\overline{b}+i\overline{c})}.$$

Denote by q' the intersection of ap and cb''. Then the points a,p,q' are colinear as well as the points b'',c,q'. Hence by T1.2

$$\frac{a-p}{\overline{a}-\overline{p}}=\frac{a-q'}{\overline{a}-\overline{q'}},\quad \frac{b''-c}{\overline{b''}-\overline{c}}=\frac{q-c}{\overline{q}-\overline{c}}.$$

The first equation gives $\overline{q'}=q'\dfrac{(1-i)\overline{b}+(1+i)\overline{c}}{(1+i)b+(1-i)c}$, and the second $\overline{q}=\dfrac{q(\overline{c}-i\overline{b})+i(\overline{b}c+b\overline{c})}{c+ib}$. By the equating we get

$$q'=\frac{(\overline{b}c+b\overline{c})((1+i)b+(1-i)c)}{(b-ic)(\overline{b}+i\overline{c})},$$

hence $q=q'$, q.e.d.

43. Assume that the origin is the intersection of the diagonals, i.e. $o=0$. From the colinearity of a,o,c and b,o,d using T1.2 we get $a\overline{c}=\overline{a}c$ and $b\overline{d}=\overline{b}d$. By T6.1 we get $m=\dfrac{a+b}{2}$ and $n=\dfrac{c+d}{2}$. Since $om\perp cd$ and $on\perp ab$ by T1.3

$$\frac{\frac{c+d}{2}-o}{\frac{\overline{c}+\overline{d}}{2}-\overline{o}}=-\frac{a-b}{\overline{a}-\overline{b}},\quad \frac{\frac{a+b}{2}-o}{\frac{\overline{a}+\overline{b}}{2}-\overline{o}}=-\frac{c-d}{\overline{c}-\overline{d}}.$$

From these two equations we get

$$c=\frac{da(\overline{a}b-2b\overline{b}+a\overline{b})}{b(\overline{a}b-2a\overline{a}+a\overline{b})}\ \text{ and } c=\frac{da(\overline{a}b+2b\overline{b}+a\overline{b})}{b(\overline{a}b+2a\overline{a}+a\overline{b})}.$$

The last two expressions give $(\overline{a}b+a\overline{b})(a\overline{a}-b\overline{b})=0$. We need to prove that the last condition is sufficient to guarantee that a,b,c,d belong to a circle. According to T3 the last is equivalent to

$$\frac{c-d}{\overline{c}-\overline{d}}\frac{b-a}{\overline{b}-\overline{a}}=\frac{b-d}{\overline{b}-\overline{d}}\frac{c-a}{\overline{c}-\overline{a}}.$$

Since the points b,d,o are colinear, by T1.2 $\dfrac{b-d}{\overline{b}-\overline{d}}=\dfrac{b-o}{\overline{b}-\overline{o}}=\dfrac{b}{\overline{b}}$ we get $\dfrac{a-c}{\overline{a}-\overline{c}}=\dfrac{a-o}{\overline{a}-\overline{o}}=\dfrac{a}{\overline{a}}$. If $a\overline{b}+\overline{a}b=0$ then

$$c-d=d\frac{2ab(\overline{a}-\overline{b})}{b(\overline{a}b-2a\overline{a}+a\overline{b})},$$

and the last can be obtained by conjugation. If $a\overline{a}=b\overline{b}$, then

$$c-d=\frac{d(a-b)(\overline{a}b+a\overline{b})}{b(\overline{a}b-2a\overline{a}+a\overline{b})},$$

and in this case we can get the desired statement by conjugation.

44. Let f be the origin and let $d=\overline{c}$ (this is possible since $FC=FD$). According to T9.2 we have that

$$o_1=\frac{ad(\overline{a}-\overline{d})}{\overline{a}d-a\overline{d}},\quad o_2=\frac{bc(\overline{b}-\overline{c})}{\overline{b}c-b\overline{c}}.$$

Since $cd\parallel af$ according to T1.1 $\dfrac{a-f}{\overline{a}-\overline{f}}=\dfrac{c-d}{\overline{c}-\overline{d}}=-1$, i.e. $\overline{a}=-a$ and similarly $\overline{b}=-b$. Now we have

$$o_1=\frac{\overline{c}(a+c)}{c+\overline{c}},\quad o_2=\frac{c(b+\overline{c})}{c+\overline{c}}.$$

Let's denote the point e. From T1.2 using the colinearity of a,c,e and b,d,e we get the following two equations

$$\frac{a-c}{\overline{a}-\overline{c}}=\frac{e-a}{\overline{e}-\overline{a}}, \quad \frac{b-d}{\overline{b}-\overline{d}}=\frac{e-b}{\overline{e}-\overline{b}}.$$

From these equations we get $\overline{e}=\dfrac{a(c+\overline{c})-e(a+\overline{c})}{a-c}$ and $\overline{e}=\dfrac{b(c+\overline{c})-e(b+c)}{b-\overline{c}}$. By equating these two we get

$$e=\frac{a\overline{c}-bc}{a+\overline{c}-b-c}.$$

Using T1.3 the condition $fe\perp o_1o_2$ is equivalent to $\dfrac{o_1-o_2}{\overline{o_1}-\overline{o_2}}=-\dfrac{f-e}{\overline{f}-\overline{e}}$, which trivially follows from $o_1-o_2=\dfrac{a\overline{c}-cb}{c+\overline{c}}$ by conjugation.

45. Assume that the point p is the origin. Let ac be the real axis and let $\angle cpd=\varphi$. Then $a=\alpha, b=\beta e^{i\varphi}, c=\gamma, d=\delta e^{i\varphi}$, where $\alpha,\beta,\gamma,\delta$ are some real numbers. Let $e^{i\varphi}=\Pi$. If $|a-f|=\varepsilon|a-d|$, then $|e-c|=\varepsilon|b-c|$ hence by T6.1 $a-f=\varepsilon(a-d)$ and $e-c=\varepsilon(b-c)$. Thus we have

$$f=\alpha(1-\varepsilon)+\varepsilon\delta\Pi,\ e=\gamma(1-\varepsilon)+\varepsilon\beta\Pi.$$

Since q belongs to pd we have that $q=\rho\Pi$ and since q also belongs to ef by T1.2 we have that $\dfrac{f-q}{\overline{f}-\overline{q}}=\dfrac{e-f}{\overline{e}-\overline{f}}$, hence

$$\frac{\alpha(1-\varepsilon)+(\varepsilon\delta-\rho)\Pi}{\alpha(1-\varepsilon)+(\varepsilon\delta-\rho)\frac{1}{\Pi}}=\frac{(1-\varepsilon)(\alpha-\gamma)+\varepsilon(\delta-\beta)\Pi}{(1-\varepsilon)(\alpha-\gamma)+\varepsilon(\delta-\beta)\frac{1}{\Pi}}.$$

After some algebra we get $(\Pi-\frac{1}{\Pi})(1-\varepsilon)\Big[(\alpha-\gamma)(\varepsilon\delta-\rho)-\varepsilon\alpha(\delta-\beta)\Big]=0$. Since $\Pi\neq\pm1$ (because $\angle CPD<180^\circ$) and $\varepsilon\neq1$ we get $\rho=\varepsilon\Big[\delta-\dfrac{\alpha(\delta-\beta)}{\alpha-\gamma}\Big]$. Similarly we get $\rho=(1-\varepsilon)\Big[\alpha-\dfrac{\delta(\alpha-\gamma)}{\delta-\beta}\Big]$, where ρ is the coordinate of the point r. By T9.2 we have

$$\begin{aligned} o_1 &= \frac{rq(\overline{r}-\overline{q})}{\overline{r}q-\overline{q}}=\frac{\rho\rho\Pi(\rho-\rho\frac{1}{\Pi})}{\rho\rho\Pi-\rho\rho\frac{1}{\Pi}}=\frac{\rho\Pi-\rho}{\Pi^2-1}\Pi \\ &= \frac{(1-\varepsilon)\Big[\alpha-\frac{\delta(\alpha-\gamma)}{\delta-\beta}\Big]\Pi-\varepsilon\Big[\delta-\frac{\alpha(\delta-\beta)}{\alpha-\gamma}\Big]}{\Pi^2-1}\Pi. \end{aligned}$$

For any other position of the point e on the line ad such that $ae=\varepsilon ad$ the corresponding center of the circle has the coordinate

$$o_2=\frac{(1-\varepsilon)\Big[\alpha-\frac{\delta(\alpha-\gamma)}{\delta-\beta}\Big]\Pi-\varepsilon\Big[\delta-\frac{\alpha(\delta-\beta)}{\alpha-\gamma}\Big]}{\Pi^2-1}\Pi.$$

Notice that the direction of the line o_1o_2 doesn't depend on ε and ε. Namely if we denote $A=\alpha-\dfrac{\delta(\alpha-\gamma)}{\delta-\beta}$ and $B=\delta-\dfrac{\alpha(\delta-\beta)}{\alpha-\gamma}$ we have

$$\frac{o_1-o_2}{\overline{o_1}-\overline{o_2}}=-\frac{A\Pi+B}{A+B\Pi}\Pi.$$

Thus for every three centers o_1, o_2, o_3 it holds $o_1o_2 \| o_2o_3$ hence all the centers are colinear. Since all the circles have a common point, the circles have another common point.
Remark. We have proved more than we've been asked. Namely two conditions $AD = BC$ and $BE = DF$ are substituted by one $BE/BC = DF/AD$.
Another advantage of this solutions is that we didn't have to guess what is the other intersection point.

46. Let o be the origin. According to the property T9.1 we have that $h_1 = \dfrac{(a-b)(\overline{a}b+a\overline{b})}{a\overline{b}-\overline{a}b}$, $h_2 = \dfrac{(c-d)(\overline{c}d+c\overline{d})}{c\overline{d}-\overline{c}d}$, and according to the theorem 6 $t_1 = \dfrac{a+c}{3}$, $t_2 = \frac{b+d}{3}$. Since the points a, c, and o are colinear as well as the points b, d, and o by T1.2 we have $\overline{c} = \dfrac{c\overline{a}}{a}$, $\overline{d} = \dfrac{d\overline{b}}{b}$, hence $h_2 = \dfrac{(c-d)(a\overline{b}+\overline{a}b)}{a\overline{b}-\overline{a}b}$. In order to prove that $t_1t_2 \perp h_1h_2$, by T1.3, it is enough to verify

$$\frac{t_1-t_2}{\overline{t_1}-\overline{t_2}} = -\frac{h_1-h_2}{\overline{h_1}-\overline{h_2}}.$$

This follows from

$$h_1 - h_2 = \frac{a\overline{b}+\overline{a}b}{a\overline{b}-\overline{a}b}\Big(a+c-b-d\Big),$$

by conjugation.

47. Let Γ be the unit circle. Using T2.3 we get $c = \dfrac{2ab}{a+b}$. Let o_1 be the center of Γ_1. Then $o_1b \perp ab$ (because ab is a tangent) hence by T1.3 $\dfrac{o_1-b}{\overline{o_1}-\overline{b}} = -\dfrac{a-b}{\overline{a}-\overline{b}} = ab$. After simplifying $\overline{o_1} = \dfrac{o_1+a-b}{ab}$. We have also $|o_1-b| = |o_1-c|$, and after squaring $(o_1-b)(\overline{o_1}-\overline{b}) = (o_1-c)(\overline{o_1}-\overline{c})$, i.e. $\overline{o_1} = \dfrac{o_1}{b^2} - \dfrac{a-b}{b(a+b)}$. Now we have

$$o_1 = \frac{ab}{a+b} + b.$$

Since the point m belongs to the unit circle it satisfies $\overline{m} = \dfrac{1}{m}$ and since it belongs to the circle with the center o_1 it satisfies $|o_1-m| = |o_1-b|$. Now we have

$$\overline{o_1}m^2 - \Big(\frac{o_1}{b} + \overline{o_1}b\Big)m + o_1 = 0.$$

This quadratic equation defines both m and b, and by Vieta's formulas we have $b+m = \dfrac{o_1}{\overline{o_1}b} + b$, i.e.

$$m = b\frac{2a+b}{a+2b}.$$

It remains to prove that the points a, m, and the midpoint of the segment bc colinear. The midpoint of bc is equal to $(b+c)/2$ by T6.1. According to T1.2 it is enough to prove that

$$\frac{a-\dfrac{b+c}{2}}{\overline{a}-\dfrac{\overline{b}+\overline{c}}{2}} = \frac{a-m}{\overline{a}-\overline{m}} = -am,$$

which is easy to verify.

48. Assume that the circle k is unit and assume that $b=1$. The $a=-1$ and since $p\in k$ we have $\overline{p}=\dfrac{1}{p}$. According to T2.4 we have that $q=\dfrac{1}{2}\Big(p+\dfrac{1}{p}\Big)$, and according to T6.1 we have that $f=\dfrac{\Big(p+\dfrac{1}{p}\Big)-1}{2}=\dfrac{(p-1)^2}{4p}$. Furthermore since c belongs to the circle with the center p and radius $|p-q|$ we have $|p-q|=|p-c|$ and after squaring

$$(p-q)(\overline{p}-\overline{q})=(p-c)(\overline{p}-\overline{c}).$$

Since $c\in k$ we have $\overline{c}=\dfrac{1}{c}$. The relation $p-q=\dfrac{1}{2}\Big(p-\dfrac{1}{p}\Big)$ implies

$$4pc^2-(p^4+6p^2+1)c+4p^3=0.$$

Notice that what we obtained is the quadratic equation for c. Since d satisfies the same conditions we used for c, then the point d is the second solution of this quadratic equation. Now from Vieta's formulas we get

$$c+d=\frac{p^4+6p^2+1}{4p^3},\quad cd=p^2.$$

Since the point g belongs to the chord cd by T2.2 we get

$$\overline{g}=\frac{c+d-g}{cd}=\frac{p^4+6p^2+1-4pg}{4p^3}.$$

From $gf\perp cd$ T1.3 gives $\dfrac{g-f}{\overline{g}-\overline{f}}=-\dfrac{c-d}{\overline{c}-\overline{d}}=cd=p^2$. Solving this system gives us

$$g=\frac{p^3+3p^2-p+1}{4p}.$$

The necessair and sufficient condition for colinearity of the points a,p,g is (according to T1.2) $\dfrac{a-g}{\overline{a}-\overline{g}}=\dfrac{a-p}{\overline{a}-\overline{p}}=p$. This easily follows from $a-g=\dfrac{p^3+3p^3+3p+1}{4p}$ and by conjugating $\overline{a}-\overline{g}=\dfrac{1+3p+3p^2+p^3}{4p^2}$. Since e belongs to the chord cd we have by T2.2 $\overline{e}=\dfrac{c+d-g}{cd}=\dfrac{p^4+6p^2+1-4pe}{4p^3}$, and since $pe\perp ab$ T1.3 implies $\dfrac{e-p}{\overline{e}-\overline{p}}=-\dfrac{a-b}{\overline{a}-\overline{b}}=-1$, or equivalently $\overline{e}=p+\frac{1}{p}-e$. It follows that $e=\dfrac{3p^2+1}{4p}$. Since $p-q=\dfrac{p^2-1}{2p}=2\dfrac{p^2-1}{4p}=2\Big(e-q\Big)$, we get $|e-p|=|e-q|$. Furthermore since $g-e=\dfrac{p^2-1}{4}$ from $|p|=1$, we also have $|e-q|=|g-e|$, which finishes the proof.

49. Assume that the circle with the diameter bc is unit and that $b=-1$. Now by T6.1 we have that $b+c=0$, i.e. $c=1$, and the origin is the midpoint of the segment bc. Since p belongs to the unit circle we have $\overline{p}=\dfrac{1}{p}$, and since $pa\perp p0$, we have according to T1.3 $\dfrac{a-p}{\overline{a}-\overline{p}}=-\dfrac{p-0}{\overline{p}-\overline{0}}=-p^2$. Simplification yields

$$\overline{a}p^2-2p+a=0.$$

Since this quadratic equation defines both p and q, according to Vieta's formulas we have

$$p+q=\frac{2}{\overline{a}},\quad pq=\frac{a}{\overline{a}}.$$

Let h' be the intersection of the perpendicular from a to bc with the line pq. Since $h' \in pq$ T2.2 gives $\overline{h'} = \frac{p+q-h'}{pq} = \frac{2-\overline{a}h}{a}$. Since $ah \perp bc$ according to T1.3 we have $\frac{a-h}{\overline{a}-\overline{h}} = -\frac{b-c}{\overline{b}-\overline{c}} = -1$, i.e. $\overline{h} = a+\overline{a}-h$. Now we get

$$h = \frac{a\overline{a}+a^2-2}{a-\overline{a}}.$$

It is enough to prove that $h' = h$, or $ch \perp ab$ which is by T1.3 equivalent to $\frac{h-c}{\overline{h}-\overline{c}} = -\frac{a-b}{\overline{a}-\overline{b}}$. The last easily follows from

$$h-1 = \frac{a\overline{a}+a^2-2-a+\overline{a}}{a-\overline{a}} = \frac{(a+1)(a+\overline{a}-2)}{a-\overline{a}}$$

and $a-b = a+1$ by conjugation.

50. Assume that the origin of our coordinate system is the intersection of the diagonals of the rectangle and that the line ab is parallel to the real axis. We have by T6.1 $c+a=0$, $d+b=0$, $c=\overline{b}$, and $d=\overline{a}$. Since the points $p,a,0$ are colinear T1.2 implies $\frac{p}{\overline{p}} = \frac{a}{\overline{a}}$, i.e. $\overline{p} = -\frac{b}{a}p$. Let $\varphi = \angle dpb = \angle pbc$. By T1.4 we have

$$\frac{c-p}{\overline{c}-\overline{p}} = e^{i2\varphi}\frac{b-p}{\overline{b}-\overline{p}}, \quad \frac{p-b}{\overline{p}-\overline{b}} = e^{i2\varphi}\frac{c-b}{\overline{c}-\overline{b}},$$

and after multiplying these equalities and expressing in terms of a and b

$$\frac{p+b}{bp+a^2} = \frac{a(p-b)^2}{(bp-a^2)^2}.$$

In the polynomial form this writes as

$$(b^2-ab)p^3+p^2(b^3-2a^2b-a^3+2ab^2)+p(a^4-2a^2b^2-ab^3+2a^3b)+a^4b-a^3b^2$$
$$=(b-a)(bp^3+(a^2+3ab+b^2)p^2-ap(a^2+3ab+b^2)-a^3b)=0.$$

Notice that a is one of those points p which satisfy the angle condition. Hence a is one of the zeroes of the polynomial. That means that p is the root of the polynomial which is obtained from the previous one after division by $p-a$ i.e. $bp^2+(a^2+3ab+b^2)p+a^2b=0$. Let's now determine the ratio $|p-b|:|p-c|$. From the previous equation we have $bp^2+a^2b = -(a^2+3ab+b^2)$, hence

$$\frac{PB^2}{PC^2} = \frac{(p-b)(\overline{p}-\overline{b})}{(p-c)(\overline{p}-\overline{c})} = \frac{bp^2-(a^2+b^2)p+a^2b}{bp^2+2abp+a^b} = \frac{-2(a^2+b^2+2ab)}{-(a^2+b^2+2ab)} = 2,$$

and the required ratio is $\sqrt{2}:1$.

51. Assume first that the quadrilateral $abcd$ is cyclic and that its cicrumcircle is the unit circle. If $\angle abd = \varphi$ and $\angle bda = \theta$ by T1.4 after squaring we have

$$\frac{d-b}{\overline{d}-\overline{b}} = e^{i2\varphi}\frac{a-b}{\overline{a}-\overline{b}}, \quad \frac{c-b}{\overline{c}-\overline{b}} = e^{i2\varphi}\frac{p-b}{\overline{p}-\overline{b}},$$
$$\frac{c-d}{\overline{c}-\overline{d}} = e^{i2\theta}\frac{p-d}{\overline{p}-\overline{d}}, \quad \frac{b-d}{\overline{b}-\overline{d}} = e^{i2\theta}\frac{a-d}{\overline{a}-\overline{d}}.$$

From the first of these equalities we get $e^{i2\varphi}\frac{a}{d}$, and from the fourth $e^{i2\theta} = \frac{b}{a}$. From the second equality we get $\overline{p} = \frac{ac+bd-pd}{abc}$, and from the third $\overline{p} = \frac{ac+bd-pb}{acd}$. Now it follows that

$$p = \frac{ac+bd}{b+d}.$$

We have to prove that $|a-p|^2=(a-p)(\overline{a}-\overline{p})=|c-p|^2=(c-p)(\overline{c}-\overline{p})$, which follows from

$$a-p=\frac{ab+ad-ac-bd}{b+d},\quad \overline{a}-\overline{p}=\frac{cd+bc-bd-ac}{ac(b+d)},$$

$$c-p=\frac{bc+cd-ac-bd}{b+d},\quad \overline{c}-\overline{p}=\frac{ad+ab-bd-ac}{ac(b+d)}.$$

Assume that $|a-p|=|c-p|$. Assume that the circumcircle of the triangle abc is unit. Squaring the last equality gives us that $a\overline{p}+\frac{p}{a}=c\overline{p}+\frac{p}{c}$, i.e. $(a-c)(\overline{p}-\frac{p}{ac})=0$. This means that $\overline{p}=\frac{p}{ac}$. Let d belong to the chord $d'c$. Then according to T2.2 $\overline{d}=\frac{c+d'-d}{cd'}$. By the condition of the problem we have $\angle dba=\angle cbp=\varphi$ and $\angle adb=\angle pdc=\theta$, and squaring in T1.4 yields

$$\frac{a-b}{\overline{a}-\overline{b}}=e^{i2\varphi}\frac{d-b}{\overline{d}-\overline{b}},\quad \frac{p-b}{\overline{p}-\overline{b}}=e^{i2\varphi}\frac{c-b}{\overline{c}-\overline{b}},$$

$$\frac{b-d}{\overline{b}-\overline{d}}=e^{i2\theta}\frac{a-d}{\overline{a}-\overline{d}},\quad \frac{c-d}{\overline{c}-\overline{d}}=e^{i2\theta}\frac{p-d}{\overline{p}-\overline{d}}.$$

Multiplying the first two equalities gives us

$$\frac{a-b}{\overline{a}-\overline{b}}\frac{c-b}{\overline{c}-\overline{b}}=ab^2c=\frac{p-b}{\overline{p}-\overline{b}}\frac{d-b}{\overline{d}-\overline{b}}.$$

After some algebra we conclude

$$p=\frac{ac+bd-b(ac\overline{d}+b)}{d-b^2\overline{d}}=\frac{bdd'+acd'-abd'-abc+abd-b^2d'}{cd'd-b^2d'+b^2d-b^2c}.$$

Since the pionts d,c,d' are colinear, according to T1.2 we get $\frac{d-c}{\overline{d}-\overline{c}}=\frac{c-d'}{\overline{c}-\overline{d'}}=-cd'$, and myltiplying the third and fourth equality gives

$$(-cd')(d-a)(\overline{d}-\overline{b})(\overline{d}-\overline{p})-(\overline{d}-\overline{a})(d-b)(d-p)=0.$$

Substituting values for p gives us a polynomial f in d. It is of the most fourth degree and observing the coefficient next to d^4 of the left and right summand we get that the polynomial is of the degree at most 3. It is obvious that a and b are two of its roots. We will now prove that its third root is d' and that would imply $d=d'$. For $d=d'$ we get

$$p=\frac{bd'd+acd'-abc-b^2d'}{c(d'^2-b^2)}=\frac{ac+bd'}{b+d'},\quad d-p=\frac{d'^2-ac}{b+d'}$$

$$\overline{d}-\overline{p}=-bd'\frac{d'^2-ac}{ac(b+d')}\quad \frac{d-a}{\overline{d}-\overline{a}}=-d'a,\quad \frac{d-b}{\overline{d}-\overline{b}}=-d'b$$

and the statement is proved. Thus $d=d'$ hence the quadrilateral $abcd'$ is cyclic.

52. Since the rectangles $a_1b_2a_2b_1$, $a_2b_3a_3b_2$, $a_3b_4a_4b_3$, and a_4,b_1,a_1,b_4 are cyclic T3 implies that the numbers

$$\frac{a_1-a_2}{b_2-a_2}:\frac{a_1-b_1}{b_2-b_1},\quad \frac{a_2-a_3}{b_3-a_3}:\frac{a_2-b_2}{b_3-b_2},$$

$$\frac{a_3-a_4}{b_4-a_4}:\frac{a_3-b_3}{b_4-b_3},\quad \frac{a_4-a_1}{b_1-a_1}:\frac{a_4-b_4}{b_1-b_4},$$

are real. The product of the first and the third divided by the product of the second and the fourth is equal to

$$\frac{a_1-a_2}{a_2-a_3}\cdot\frac{a_3-a_4}{a_4-a_1}\cdot\frac{b_2-b_1}{b_3-b_2}\cdot\frac{b_4-b_3}{b_1-b_4},$$

and since the points a_1,a_2,a_3,a_4 lie on a circle according to the theorem 4 the number $\frac{a_1-a_2}{a_2-a_3}\cdot\frac{a_3-a_4}{a_4-a_1}$ is real, hence the number $\frac{b_2-b_1}{b_3-b_2}\cdot\frac{b_4-b_3}{b_1-b_4}$ is real as well. According to T3 the points b_1,b_2,b_3,b_4 are cyclic or colinear.

53. Assume that the origin is the intersection of the diagonals of the parallelogram. Then $c=-a$ and $d=-b$. Since the triangles cde and fbc are similar and equally orientged by T4

$$\frac{c-b}{b-f}=\frac{e-d}{d-c},$$

hence $f=\frac{be+c^2-bc-cd}{e-d}=\frac{be+a^2}{e+b}$. In order for triangles cde and fae to be similar and equally oriented (as well as for fbc and fae), according to T4 it is necessairy and sufficient that the following relation holds:

$$\frac{c-d}{d-e}=\frac{f-a}{a-e}.$$

The last equaliy follows from

$$f-a=\frac{be+a^2-ea-ab}{e+b}=\frac{(e-a)(b-a)}{e+b},$$

and $c-d=c+b, d-e=-(b+e), c+b=b-a$.

54. Let $p=0$ and $q=1$. Since $\angle mpq=\alpha$, according to T1.4 we have that $\frac{q-p}{\overline{q}-\overline{p}}=e^{i2\alpha}\frac{m-p}{\overline{m}-\overline{p}}$, i.e. $\frac{m}{\overline{m}}=e^{i2\alpha}$. Since $\angle pqm=\beta$, the same theorem implies $\frac{m-q}{\overline{m}-\overline{q}}=e^{i2\beta}\frac{p-q}{\overline{p}-\overline{q}}$, i.e. $1=e^{i2\beta}\frac{m-1}{\overline{m}-1}$. Solving this system (with the aid of $e^{i2(\alpha+\beta+\gamma)}=1$) we get $m=\frac{e^{i2(\alpha+\gamma)}-1}{e^{i2\gamma}-1}$, and symmetrically $l=\frac{e^{i2(\beta+\gamma)}-1}{e^{i2\beta}-1}$, $k=\frac{e^{i2(\alpha+\beta)}-1}{e^{i2\alpha}-1}$. According to T4 in order to prove that the triangles klm and kpq are similar and equally oriented it is enough to prove that $\frac{k-l}{l-m}=\frac{k-p}{p-q}=-k$. The last follows from

$$\begin{aligned}\frac{k-l}{l-m} &= \frac{\dfrac{e^{i(2\alpha+4\beta)}-e^{i2\beta}-e^{i(2\alpha+2\beta)}+e^{i(2\beta+2\gamma)}+e^{i2\alpha}-1}{(e^{i2\alpha}-1)(e^{i2\beta}-1)}}{\dfrac{e^{i(2\beta+4\gamma)}-e^{i2\gamma}-e^{i(2\beta+2\gamma)}+e^{i(2\alpha+2\gamma)}+e^{i2\beta}-1}{(e^{i2\beta}-1)(e^{i2\gamma}-1)}}\\ &= \frac{e^{i2(\alpha+\beta)}(e^{i(2\beta+4\gamma)}-e^{i2\gamma}-e^{i(2\beta+2\gamma)}+e^{i(2\alpha+2\gamma)}+e^{i2\beta}-1)}{e^{i(2\beta+4\gamma)}-e^{i2\gamma}-e^{i(2\beta+2\gamma)}+e^{i(2\alpha+2\gamma)}+e^{i2\beta}-1}\cdot\\ &\quad \frac{e^{i2\gamma}-1}{e^{i2\alpha}-1}\\ &= \frac{1-e^{i2(\alpha+\beta)}}{e^{i2\alpha}-1}=-k.\end{aligned}$$

Since the triangles kpq, qlp, pqm are mutually similar and equally oriented the same holds for all four of the triangles.

55. Assume that the coordinates of the vertices of the i-th polygon are denoted by $a_1^{(i)}, a_2^{(i)}, \dots, a_n^{(i)}$, respectively in positive direction. smeru. According to T6.1 and the given recurrent relation we have that for each i and k:

$$a_i^{(k+1)} = 2a_{i+k}^{(k)} - a_i^{(k)},$$

where the indices are modulo n. Our goal is to determine the value of $a_i^{(n)}$, using the values of $a_1^{(1)}, a_2^{(1)}, \dots, a_n^{(1)}$. The following

$$\begin{aligned} a_i^{(k+1)} &= 2a_{i+k}^{(k)} - a_k^{(i)} = 4a_{i+k+k-1}^{(k-1)} - 2a_{i+k}^{(k-1)} - 2a_{i+k-1}^{(k-1)} + a_i^{(k-1)} \\ &= 4(2a_{i+k+k-1+k-2}^{(k-2)} - -a_{i+k+k-1}^{(k-2)}) - 2(2a_{i+k+k-2}^{(k-2)} - a_{i+k}^{(k-2)}) - \\ &\quad 2(2a_{i+k-1+k-2}^{(k-2)} - a_{i+k-1}^{(k-2)}) + 2a_{i+k-2}^{(k-2)} - a_i^{(k-2)} \\ &= 8a_{i+k+k-1+k-2}^{(k-2)} - 4(a_{i+k+k-1}^{(k-2)} + a_{i+k+k-2}^{(k-2)} + a_{i+k-1+k-2}^{(k-2)}) + \\ &\quad 2(a_{i+k}^{(k-2)} + a_{i+k-1}^{(k-2)} + a_{i+k-2}^{(k-2)}) - a_i^{(k-2)}, \end{aligned}$$

yields that

$$a_i^{(k)} = 2^{k-1} s_k^{(k)}(i) - 2^{k-2} s_{k-1}^{(k)}(i) + \dots + (-1)^k s_0^{(k)}(i),$$

where $s_j^{(k)}(i)$ denotes the sum of all the numbers of the form $a_{i+s_k(j)}$ and $s_k(j)$ is one of the numbers obtained as the sum of exactly j different natural numbers not greater than n. Here we assume that $s_0^{(k)}(i) = a_i$. The last formula is easy to prove by induction. Particularly, the formula holds for $k = n$ hence

$$a_i^{(n)} = 2^{n-1} s_n^{(n)}(i) - 2^{n-2} s_{n-1}^{(n)}(i) + \dots + (-1)^n s_0^{(n)}(i).$$

Now it is possible to prove that $s_l^{(n)}(i) = s_l^{(n)}(j)$, for each $1 \le l \le n-1$ which is not very difficult problem in the number theory. Since n is prime we have that $n + n - 1 + \dots + 1$ is divisible by n hence

$$\begin{aligned} a_i^{(n)} - a_j^{(n)} &= 2^{n-1} a_{i+n+n-1+\dots+1}^{(1)} - 2^{n-1} a_{j+n+n-1+\dots+1}^{(1)} + \\ &\quad (-1)^n a_i^{(1)} - (-1)^n a_j^{(1)} \\ &= (2^{n-1} + (-1)^n)(a_i^{(1)} - a_j^{(1)}), \end{aligned}$$

which by T4 finishes the proof.

56. Assume that the pentagon $abcde$ is inscribed in the unit circle and that x, y, and z are feet of perpendiculars from a to bc, cd, and de respectively. According to T2.4 we have that

$$x = \frac{1}{2}\left(a + b + c - \frac{bc}{a}\right), \quad y = \frac{1}{2}\left(a + c + d - \frac{cd}{a}\right), \quad z = \frac{1}{2}\left(a + d + e - \frac{de}{a}\right),$$

and according to T5 we have

$$S(xyz) = \frac{i}{4} \begin{vmatrix} x & \overline{x} & 1 \\ y & \overline{y} & 1 \\ z & \overline{z} & 1 \end{vmatrix} = \frac{i}{8} \begin{vmatrix} a+b+c-\frac{bc}{a} & \overline{a}+\overline{b}+\overline{c}-\frac{\overline{bc}}{\overline{a}} & 1 \\ a+c+d-\frac{cd}{a} & \overline{a}+\overline{c}+\overline{d}-\frac{\overline{cd}}{\overline{a}} & 1 \\ a+d+e-\frac{de}{a} & \overline{a}+\overline{d}+\overline{e}-\frac{\overline{de}}{\overline{a}} & 1 \end{vmatrix}.$$

Since the determinant is unchanged after substracting some columns from the others, we can substract the second column from the third, and the first from the second. After that we get

$$\begin{aligned} S(xyz) &= \frac{i}{8}\begin{vmatrix} a+b+c-\frac{bc}{a} & \overline{a}+\overline{b}+\overline{c}-\frac{\overline{b}\overline{c}}{\overline{a}} & 1\\ \frac{(d-b)(a-c)}{a} & \frac{(d-b)(a-c)}{bcd} & 0\\ \frac{(e-c)(a-d)}{a} & \frac{(e-c)(a-d)}{a} & 0\end{vmatrix}\\ &= \frac{i(a-c)(d-b)(a-d)(e-c)}{8}\cdot\\ &\begin{vmatrix} a+b+c-\frac{bc}{a} & \overline{a}+\overline{b}+\overline{c}-\frac{\overline{b}\overline{c}}{\overline{a}} & 1\\ \frac{1}{a} & \frac{1}{bcd} & 0\\ \frac{1}{a} & \frac{1}{a} & 0\end{vmatrix},\end{aligned}$$

and finally

$$\begin{aligned} S(xyz) &= \frac{i(a-c)(d-b)(a-d)(e-c)}{8}\Big(\frac{1}{acde}-\frac{1}{abcd}\Big)\\ &= \frac{i(a-c)(d-b)(a-d)(e-c)(b-e)}{8abcde}.\end{aligned}$$

Since the last expression is symmetric with respet to a,b,c,d, and e the given area doesn't depend on the choice of the vertex (in this case a).

57. Assume that the unit circle is the circumcircle of the triangle abc. Since $\frac{S(bca_1)}{S(abc)} = 1 - \frac{|a-a_1|}{|a-a'|} = 1-\frac{a-a_1}{a-a'}$ (where a' is the foot of the perpendicular from a to bc), the given equality becomes

$$2 = \frac{a-a_1}{a-a'}+\frac{b-b_1}{b-b'}+\frac{c-c_1}{c-c'}.$$

According to T2.4 we have $a' = \frac{1}{2}\Big(a+b+c-\frac{bc}{a}\Big)$, hence

$$a-a' = \frac{1}{2}\Big(a+\frac{bc}{a}-b-c\Big) = \frac{(a-b)(a-c)}{2a}$$

and after writing the symmetric expressions we get

$$\begin{aligned} 2 &= \frac{2a(a-a_1)}{(a-b)(a-c)}+\frac{2b(b-b_1)}{(b-a)(b-c)}+\frac{2c(c-c_1)}{(c-a)(c-b)}\\ &= -2\frac{a(a-a_1)(b-c)+b(b-b_1)(c-a)+c(c-c_1)(a-b)}{(a-b)(b-c)(c-a)},\end{aligned}$$

and after simplying

$$aa_1(b-c)+bb_1(c-a)+cc_1(a-b) = 0.$$

By T4 points a_1,b_1,c_1,h lie on a circle if and only if

$$\frac{a_1-c_1}{\overline{a_1}-\overline{c_1}}\frac{b_1-h}{\overline{b_1}-\overline{h}} = \frac{a_1-h}{\overline{a_1}-\overline{h}}\frac{b_1-c_1}{\overline{b_1}-\overline{c_1}}.$$

Since h is the orthocenter by T6.3 we have $h=a+b+c$, and since $aa_1 \perp bc$ T1.3 implies $\dfrac{a_1-a}{\overline{a_1}-\overline{a}} = -\dfrac{b-c}{\overline{b}-\overline{c}}$, i.e. $\overline{a_1} = \dfrac{bc+aa_1-a^2}{abc}$, and symmetrically $\overline{b_1} = \dfrac{ac+bb_1-b^2}{abc}$ and $\overline{c_1} = \dfrac{ab+cc_1-c^2}{abc}$. Similarly from $a_1h \perp bc$ and $b_1h \perp ac$

$$\frac{a_1-h}{\overline{a_1}-\overline{h}} = -\frac{b-c}{\overline{b}-\overline{c}} = bc, \quad \frac{b_1-h}{\overline{b_1}-\overline{h}} = -\frac{a-c}{\overline{a}-\overline{c}} = ac.$$

It is enough to prove that

$$\frac{a(a_1-c_1)}{aa_1-cc_1+(c-a)(a+b+c)} = \frac{b(b_1-c_1)}{bb_1-cc_1+(c-b)(a+b+c)}.$$

Notice that

$$a(b-c)a_1-a(b-c)c_1 = -b_1b(c-a)a-cc_1(a-b)a-a(b-c)c_1 = ab(c-a)(c_1-b_1),$$

and the result follows by the conjugation.

58. Assume that the unit circle is the circumcircle of the triangle abc. By T2.4 we have that $d = \dfrac{1}{2}\Big(a+b+c-\dfrac{ab}{c}\Big)$, $e=\dfrac{1}{2}\Big(a+b+c-\dfrac{ac}{b}\Big)$, and $f=\dfrac{1}{2}\Big(a+b+c-\dfrac{bc}{a}\Big)$. According to T6.1 we get $a_1 = \dfrac{b+c}{2}$ (where a_1 is the midpoint of the sidebc). Since q belongs to the chord ac T2.2 implies $\overline{q} = \dfrac{a+c-q}{ac}$, and since $qd \parallel ef$ T1.1 implies $\dfrac{q-d}{\overline{q}-\overline{d}} = \dfrac{e-f}{\overline{e}-\overline{f}} = -a^2$. Solving this system gives us

$$q = \frac{a^3+a^2b+abc-b^2c}{2ab}.$$

Symmetrically we get $r = \dfrac{a^3+a^2c+abc-bc^2}{2ac}$. Since p belongs to the chord bc T2.2 implies $\overline{p} = \dfrac{b+c-p}{bc}$, and from the colinearity of the points e, f, and p from T1.2 we conclude $\dfrac{p-e}{\overline{p}-\overline{e}} = \dfrac{e-f}{\overline{e}-\overline{f}} = -a^2$. After solving this system we get

$$p = \frac{a^2b+a^2c+ab^2+ac^2-b^2c-bc^2-2abc}{2(a^2-bc)} = \frac{b+c}{2}+\frac{a(b-c)^2}{2(a^2-bc)}.$$

By T4 it is sufficient to prove that

$$\frac{p-a_1}{p-r}\frac{q-r}{q-a_1} = \frac{\overline{p}-\overline{a_1}}{\overline{p}-\overline{r}}\frac{\overline{q}-\overline{r}}{\overline{q}-\overline{a_1}}.$$

Since

$$q-r = \frac{a(c-b)(a^2+bc)}{2abc}, \quad p-a_1 = \frac{a(b-c)^2}{2(a^2-bc)},$$

$$p-r = \frac{(a^2-c^2)(b^2c+abc-a^3-a^2c)}{2ac(a^2-bc)}, \quad q-a_1 = \frac{a^3+a^2b-b^2c-ab^2}{2ab}$$

the required statement follows by conjugation.

59. Let O be the circumcenter of the triangle abc. We will prove that O is the incenter as well. Assume that the circumcircle of the triangle abc is unit. According to T6.1 we have that $c_1 = \dfrac{a+b}{2}$,

$b_1 = \frac{a+c}{2}$, and $a_1 = \frac{b+c}{2}$. Assume that k_1, k_2, k_3 are the given circles with the centers a_1, b_1, and c_1. Let $k_1 \cap k_2 = \{k, o\}$, $k_2 \cap k_3 = \{m, o\}$, and $k_3 \cap k_1 = \{l, o\}$. Then we have $|a_1 - k| = |a_1 - o|$, $|b_1 - k| = |b_1 - o|$. After squaring $(a_1 - k)(\overline{a_1} - \overline{k}) = a_1\overline{a_1}$ and $(b_1 - k)(\overline{b_1} - \overline{k}) = b_1\overline{b_1}$. After solving this system we obtain

$$k = \frac{(a+c)(b+c)}{2c}.$$

Symmetrically we get $l = \frac{(b+c)(a+b)}{2b}$ and $m = \frac{(a+c)(a+b)}{2a}$. Let $\angle mko = \varphi$. According to T1.4 we have that $\frac{o-k}{\overline{o}-\overline{k}} = e^{i2\varphi}\frac{m-k}{\overline{m}-\overline{k}}$, and since $k - m = \frac{b(a^2-c^2)}{2ac}$, after conjugation $e^{i2\varphi} = -\frac{a}{b}$. If $\angle okl = \psi$, we have by T1.4 $\frac{o-k}{\overline{o}-\overline{k}} = e^{i2\psi}\frac{l-k}{\overline{l}-\overline{k}}$, hence $e^{i\psi} = -\frac{a}{b}$. Now we have $\varphi = \psi$ or $\varphi = \psi \pm \pi$, and since the second condition is impossible (why?), we have $\varphi = \psi$. Now it is clear that o is the incenter of the triangle klm.

For the second part of the problem assume that the circle is inscribed in the triangle klm is the unit circle and assume it touches the sides kl, km, lm at u, v, w respectively. According to T7.1 we have that

$$k = \frac{2uv}{u+v}, \quad l = \frac{2uw}{w+u}, \quad m = \frac{2vw}{v+w}.$$

Let a_1 be the circumcenter of the triangle kol. Then according to T9.2 we have

$$a_1 = \frac{kl(\overline{k}-\overline{l})}{\overline{k}l - k\overline{l}} = \frac{2uvw}{k(u+v)(u+w)}$$

and symmetrically $b_1 = \frac{2uvw}{(u+v)(v+w)}$ and $c_1 = \frac{2uvw}{(w+u)(w+v)}$ (b_1 and c_1 are circumcenters of the triangles kom and mol respectively). Now T6.1 implies

$$a+b = 2c_1, \quad b+c = 2a_1, \quad a+c = 2b_1,$$

and after solving this system we get $a = b_1 + c_1 - a_1$, $b = a_1 + c_1 - b_1$, and $c = a - 1 + b_1 - c_1$. In order to finish the proof it is enough to establish $ab \perp oc_1$ (the other can be proved symmetrically), i.e. by T1.3 that $\frac{c_1 - o}{\overline{c_1} - \overline{o}} = -\frac{a-b}{\overline{a}-\overline{b}} = -\frac{b_1 - a_1}{\overline{b_1} - \overline{a_1}}$. The last easily follows from

$$b_1 - a_1 = \frac{2uvw(u-v)}{(u+v)(v+w)(w+u)},$$

by conjugation.

60. Let b and c be the centers of the circles k_1 and k_2 respectively and assume that bc is the real axis. If the points m_1 and m_2 move in the same direction using T1.4 we get that m_1 and m_2 satisfy

$$m_1 - b = (a-b)e^{i\varphi}, \quad m_2 - c = (a-c)e^{i\varphi}.$$

If ω is the requested point, we must have $|\omega - m_1| = |\omega - m_2|$, and after squaring $(\omega - m_1)(\overline{\omega} - \overline{m_1}) = (\omega - m_2)(\overline{\omega} - \overline{m_2})$. From the last equation we get

$$\overline{\omega} = \frac{m_1\overline{m_1} - m_2\overline{m_2} - \omega(\overline{m_1} - \overline{m_2})}{m_1 - m_2}.$$

After simplification (with the usage of $\overline{b} = b$ and $\overline{c} = c$ where $e^{i\varphi} = z$)

$$\overline{w}(1-z) = 2(b+c) - a - \overline{a} + az + \overline{a}\overline{z} - (b+c)(z+\overline{z}) - (1-\overline{z})\omega.$$

Since $\overline{z} = \dfrac{1}{z}$, we have

$$(b+c-a-\overline{w})z^2-(2(b+c)-a-\overline{a}-\omega-\overline{\omega})z+b+c-\overline{a}-\omega \equiv 0.$$

The last polynomial has to be identical to 0 hence each of its coefficients is 0, i.e. $\omega = b+c-\overline{a}$. From the previous relations we conclude that this point satisfies the conditions of the problem. The problem is almost identical in the case of the oposite orientition.

61. Let γ be the unit circle and let $a=-1$. Then $b=1$, $c=1+2i$, and $d=-1+2i$. Since the points n,b,p are colinear we can use T1.2 to get

$$\frac{a-p}{\overline{a}-\overline{p}} = \frac{a-m}{\overline{a}-\overline{m}} = -am = m,$$

and after some algebra $\overline{p} = \dfrac{p+1-m}{m}$ (1). Since the points c,d,p are colinear using the same argument we get that

$$\frac{c-n}{\overline{c}-\overline{n}} = \frac{c-d}{\overline{c}-\overline{d}} = 1,$$

hence $\overline{p} = p-4i$. Comparing this with (1) one gets $p = 4i \cdot \dfrac{m}{m-1} - 1$. Furthermore, since the points b,n,p are colinear we have

$$\frac{p-1}{\overline{p}-\overline{1}} = \frac{1-n}{\overline{1}-\overline{n}} = n,$$

i.e.

$$n = \frac{m(1-2i)-1}{2i+1-m}.$$

Let q' be the intersection point of the circle γ and the line dm. If we show that the points q',n,c are colinear we would have $q=q'$ and $q \in \gamma$, which will finish the first part of the problem. Thus our goal is to find the coordinate of the point q'. Since q' belongs to the unit circle we have $q'\overline{q'} = 1$, and since d,m,q' are colinear, we have using T1.2 that

$$\frac{d-m}{\overline{d}-\overline{m}} = \frac{q'-m}{\overline{q'}-\overline{m}} = -q'm,$$

and after simplification

$$q' = -\frac{m+1-2i}{m(1+2i)+1}.$$

In order to prove that the points q',n,c are colinear it suffices to show that $\dfrac{q-c}{\overline{q}-\overline{c}} = \dfrac{n-q}{\overline{n}-\overline{q}} = -nq$, i.e. $n = \dfrac{q-1-2i}{(\overline{q}-1+2i)q}$, which is easy to verify. This proves the first part of the problem.

Now we are proving the second part. Notice that the required inequality is equivalent to $|q-a| \cdot |p-c| = |d-p| \cdot |b-q|$. From the previously computed values for p and q, we easily obtain

$$|q-a| = 2\left|\frac{m+1}{m(1+2i)+1}\right|, \ |p-c| = 2\left|\frac{m(1+i)+1-i}{m(1+2i)+1}\right|,$$

$$|d-p| = 2\left|\frac{m+1}{m+1}\right|, \ |b-q| = 2\left|\frac{m(i-1)+1+1}{m-1}\right|,$$

and since $-i((i-1)m+1+i) = m(1+i)+1-i$ the required equality obviously holds.

62. In this problem we have plenty of possibilities for choosing the unit circle. The most convenient choice is the circumcircle of $bcb'c'$ (try if you don't believe). According T2.5 we have that the intersection point x of bb' and cc' satisfy

$$x=\frac{bb'(c+c')-cc'(b+b')}{bb'-cc'}.$$

Since $bh\perp cb'$ and $ch\perp bc'$ T1.3 implies the following two equalities

$$\frac{b-h}{\overline{b}-\overline{h}}=-\frac{b'-c}{\overline{b'}-\overline{c}}=b'c,\quad \frac{c-h}{\overline{c}-\overline{h}}=-\frac{b-c'}{\overline{b}-\overline{c'}}=bc'.$$

From the first we get $\overline{h}=\dfrac{bh-b^2+b'c}{bb'c}$, and from the second $\overline{h}=\dfrac{ch-c^2+bc'}{bcc'}$. After equating the two relations we get

$$h=\frac{b'c'(b-c)+b^2c'-b'c^2}{bc'-b'c}.$$

Symmetrically we obtain $h'=\dfrac{bc(b'-c')+b'^2c-bc'^2}{b'c-bc'}$. It suffices to prove that the points h,h' and x are colinear, or after applying T1.2 we have to verify

$$\frac{h-h'}{\overline{h}-\overline{h'}}=\frac{h-x}{\overline{h}-\overline{x}}.$$

The last follwos from

$$\begin{aligned} h-h' &= \frac{bc(b'-c')+b'c'(b-c)+bc'(b-c')+b'c(b'-c)}{bc'-b'c}\\ &= \frac{(b+b'-c-c')(bc'+b'c)}{bc'-b'c},\\ h-x &= \frac{b^2b'^2c'+b^3b'c'+b'c^2c'^2+b'c^3c'}{(bc'-b'c)(bb'-cc')}-\\ &\quad \frac{b^2b'cc'+b^2b'c'^2+bb'c^2c'+b'^2c^2c'}{(bc'-b'c)(bb'-cc')}\\ &= \frac{b'c'(b^2-c^2)(b'+b-c-c')}{(bc'-b'c)(bb'-cc')}\end{aligned}$$

by conjugation.

63. From elementary geometry we know that $\angle nca=\angle mcb$ (such points m and n are called harmonic conjugates). Let $\angle mab=\alpha$, $\angle abm=\beta$, and $\angle mca=\gamma$. By T1.4 we have that

$$\frac{a-b}{|a-b|}=e^{i\alpha}\frac{a-m}{|a-m|},\ \frac{a-n}{|a-n|}=e^{i\alpha}\frac{a-c}{|a-c|},$$
$$\frac{b-c}{|b-c|}=e^{i\beta}\frac{b-n}{|b-n|},\ \frac{b-m}{|b-m|}=e^{i\beta}\frac{b-a}{|b-a|},$$
$$\frac{c-a}{|c-a|}=e^{i\gamma}\frac{c-n}{|c-n|},\quad \frac{c-m}{|c-m|}=e^{i\gamma}\frac{c-b}{|c-b|},$$

hence

$$\begin{aligned}&\frac{AM\cdot AN}{AB\cdot AC}+\frac{BM\cdot BN}{BA\cdot BC}+\frac{CM\cdot CN}{CA\cdot CB}\\ =\ &\frac{(m-a)(n-a)}{(a-b)(a-c)}+\frac{(m-b)(n-b)}{(b-a)(b-c)}+\frac{(m-c)(n-c)}{(c-a)(c-b)}.\end{aligned}$$

The last expression is always equal to 1 which finishes our proof.

64. Let $\angle A = \alpha$, $\angle B = \beta$, $\angle C = \gamma$, $\angle D = \delta$, $\angle E = \varepsilon$, and $\angle F = \varphi$. Applying T1.4 gives us

$$\frac{b-c}{|b-c|} = e^{i\beta}\frac{b-a}{|b-a|}, \quad \frac{d-e}{|d-e|} = e^{i\delta}\frac{d-c}{|d-c|}, \quad \frac{f-a}{|f-a|} = e^{i\varphi}\frac{f-e}{|f-e|}.$$

Multiplying these equalities and using the given conditions (from the conditions of the problem we read $e^{i(\beta+\delta+\varphi)} = 1$) we get

$$(b-c)(d-e)(f-a) = (b-a)(d-c)(f-e).$$

From here we can immediately conclude that

$$(b-c)(a-e)(f-d) = (c-a)(e-f)(d-b),$$

and the result follows by placing the modulus in the last expression.

65. We first apply the inversion with repsect to the circle ω. The points a,b,c,e,z are fixed, and the point d is mapped to the intersection of the lines ae and bc. Denote that intersection by s. The circumcircle of the triangle azd is mapped to the circumcircle of the triangle azs, the line bd is mapped to the line bd, hence it is sufficient to prove that bd is the tangent to the circle circumscribed about azs. The last is equivalent to $az \perp sz$.

Let ω be the unit circle and let $b=1$. According to T6.1 we have $c=-1$ and $e=\overline{a}=\frac{1}{a}$. We also have $s=\frac{a+\overline{a}}{2}=\frac{a^2+1}{2a}$. Since $eb \perp ax$ using T1.3 we get

$$\frac{a-x}{\overline{a}-\overline{x}} = -\frac{e-b}{\overline{e}-\overline{b}} = -\frac{1}{a},$$

and since the point x belongs to the chord eb by T2.2 it satisfies $\overline{x}=\frac{1+\overline{a}-x}{\overline{a}}$. Solving this system gives sistema dobijamo $x=\frac{a^3+a^2+a-1}{2a^2}$. Since y is the midpoint of ax by T6.1

$$y=\frac{a+x}{2}=\frac{3a^3+a^2+a-1}{4a^2}.$$

Since the points b,y,z are colinear and z belongs to the unit circle according to T1.2 and T2.1 we get

$$\frac{b-y}{\overline{b}-\overline{y}} = \frac{b-z}{\overline{b}-\overline{z}} = -z.$$

After simplifying we get $z=\frac{1+3a^2}{(3+a^2)a}$. In order to prove that $az \perp zs$ by T1.3 it is sufficient to prove that

$$\frac{a-z}{\overline{a}-\overline{z}} = -\frac{s-z}{\overline{s}-\overline{z}}.$$

The last follows from

$$a-z=\frac{a^4-1}{a(3+a^2)}, \quad s-z=\frac{a^4-2a^2+1}{2a(3+a^2)},$$

by conjugation.

66. Assume first that the orthocenters of the given triangles coincide. Assume that the circumcircle of abc is unit. According to T6.3 we have $h=a+b+c$. Consider the rotation with respect to h

for the angle ω in the negative direction. The point a_1 goes to the point a_1' such that a_1, a_1', and h are colinear. Assume that the same rotation maps b_1 to b_1' and c_1 to c_1'. Since the triangles abc and $a_1b_1c_1$ are similar and equally oriented we get that the points b, b_1', h are clinear as well as c, c_1', h. Moreover $a_1'b_1' \parallel ab$ (and similarly for $b_1'c_1'$ and $c_1'a_1'$). Now according to T1.4 $e^{i\omega}(a_1' - h) = (a_1 - h)$ (since the rotation is in the negative direction), and since the points a, a_1', h are colinear, according to T1.2 we have $\dfrac{a_1' - h}{a - h} = \lambda \in \mathbf{R}$. This means that $a_1 = h + \lambda e^{i\omega}(a - h)$ and analogously

$$b_1 = h + \lambda e^{i\omega}(b - h), \quad c_1 = h + \lambda e^{i\omega}(c - h).$$

Since the point a_1 belongs to the chord bc of the unit circle, by T2.2 we get $\overline{a_1} = \dfrac{b + c - a_1}{bc}$. On the other hand by conjugation of the previous expression for a_1 we get $\overline{a_1} = \overline{h} + \lambda \dfrac{\overline{a} - \overline{h}}{e^{i\omega}}$. Solving for λ gives

$$\lambda = \frac{e^{i\omega}(a(a+b+c)+bc)}{a(b+c)(e^{ie\omega}+1)}. \quad (1)$$

Since λ has the same role in the formulas for b_1 also, we must also have

$$\lambda = \frac{e^{i\omega}(b(a+b+c)+ac)}{b(a+c)(e^{ie\omega}+1)}. \quad (2)$$

By equating (1) and (2) we get

$$\begin{aligned} & ab(a+c)(a+b+c) + b^2c(a+c) - ab(b+c)(a+b+c) - a^2c(b+c) \\ = \ & (a-b)(ab(a+b+c) - abc - ac^2 - bc^2) = (a^2 - b^2)(ab - c^2). \end{aligned}$$

Since $a^2 \neq b^2$ we conclude $ab = c^2$. Now we will prove that this is necessair condition for triangle abc to be equilateral, i.e. $|a - b| = |a - c|$. After squaring the last expression we get that the triangle is equilateral if and only if $0 = \dfrac{(a-c)^2}{ac} - \dfrac{(a-b)^2}{ab} = \dfrac{(b-c)(a^2-bc)}{abc}$, and since $b \neq c$, this part of the problem is solved.

Assume now that the incenters of the given triangles coincide. Assume that the incircle of the triangle abc is unit and let d, e, f be the points of tangency of the incircle with the sides ab, bc, ca respectively. Similarly to the previous part of the problem we prove

$$a_1 = i + \lambda e^{i\omega}(a - i), \quad b_1 = i + \lambda e^{i\omega}(b - i), \quad c_1 = i + \lambda e^{i\omega}(c - i).$$

Together with the condition $i = 0$ T2.3 and conjugation imply $\overline{a_1} = \dfrac{2\lambda}{e^{i\omega}(e+f)}$. Also, since the points a_1, b, c are colinear we have $a_1d \perp di$ hence according to T1.3 $\dfrac{a_1 - d}{\overline{a_1} - \overline{d}} = -\dfrac{d - i}{\overline{d} - \overline{i}} = -d^2$. Solving this system gives

$$\lambda = \frac{d(e+f)}{d^2 + efe^{i\omega}}.$$

Since λ has the same roles in the formulas for a_1 and b_1 we must have

$$\lambda = \frac{e(d+f)}{e^2 + dfe^{i\omega}},$$

and equating gives us

$$e^{i2\omega} = \frac{ed(e+d+f)}{f(de+ef+fd)}.$$

Symmetry implies $e^{i2\omega}=\dfrac{ef(e+d+f)}{d(de+ef+fd)}$ and since $f^2\neq d^2$ we must have $e+d+f=0$. It is easy to prove that the triangle def is equilateral in this case as well as abc.

67. Since $(a-b)(c-d)+(b-c)(a-d)=(a-c)(b-d)$ the triangle inequality implies $|(a-b)(c-d)|+|(b-c)(a-d)|\geq|(a-c)(b-d)|$, which is exactly an expression of the required inequality. The equality holds if and only if the vectors $(a-b)(c-d)$, $(b-c)(a-d)$, and $(a-c)(c-d)$ are colinear. The first two of them are colinear if and only if

$$\frac{(a-b)(c-d)}{(b-c)(a-d)}\in\mathbf{R},$$

which is according to T3 precisely the condition that a,c,b,d belong to a circle. Similarly we prove that the other two vectors are colinear.

68. Since $(d-a)(d-b)(a-b)+(d-b)(d-c)(b-c)+(d-c)(d-a)(c-a)=(a-b)(b-c)(c-a)$, we have $|(d-a)(d-b)(a-b)|+|(d-b)(d-c)(b-c)|+|(d-c)(d-a)(c-a)|\geq|(a-b)(b-c)(c-a)|$ where the equality holds if and only if $(d-a)(d-b)(a-b),(d-b)(d-c)(b-c),(d-c)(d-a)(c-a)$ and $(a-b)(b-c)(c-a)$ are colinear. The condition for colinearity of the first two vectors can be expressed as

$$\frac{(d-a)(a-b)}{(d-c)(b-c)}=\frac{(\overline{d}-\overline{a})(\overline{a}-\overline{b})}{(\overline{d}-\overline{c})(\overline{b}-\overline{c})}.$$

Assume that the circumcircle of abc is unit. Now the given expression can be written as

$$d\overline{d}a-a^2\overline{d}-\frac{da}{c}+\frac{a^2}{c}=d\overline{d}c-c^2\overline{d}-\frac{dc}{a}+\frac{c^2}{a}$$

and after some algebra $d\overline{d}(a-c)=(a-c)\Big((a+c)\Big(\overline{d}+\dfrac{d}{ac}-\dfrac{a+c}{ac}\Big)+1\Big)$ or

$$d\overline{d}=(a+c)\Big(\overline{d}+\frac{d}{ac}-\frac{a+c}{ac}\Big)+1.$$

Similarly, from the colinearity of the first and the third vector we get $d\overline{d}=(b+c)\Big(\overline{d}+\dfrac{d}{bc}-\dfrac{b+c}{bc}\Big)+1$. Substracting the last two expressions yields $(a-b)\Big(\overline{d}-\dfrac{d}{ab}+\dfrac{c^2-ab}{abc}\Big)=0$, i.e.

$$\overline{d}-\frac{d}{ab}+\frac{c^2-ab}{abc}=0.$$

Similarly $\overline{d}-\dfrac{d}{ac}+\dfrac{b^2-ac}{abc}=0$ and after substracting and simplifying we get $d=a+b+c$. It is easy to verify that for $d=a+b+c$, i.e. the orthocenter of the triangle abc, all four of the above mentioned vectors colinear.

13 Problems for Indepent Study

For those who want more, here is the more. Many of the following problems are similar to the problems that are solved above. There are several quite difficult problems (towards the end of the list) which require more attention in choosing the known points, and more time. As in the case with solved problems, I tried to put lot of problems from math competitions from all over the world.

1. (Regional competition 2002, 2nd grade) In the acute-angled triangle ABC, B' and C' are feet of perpendiculars from the vertices B and C respectively. The circle with the diameter AB intersects the line CC' at the points M and N, and the circle with the diameter AC intersects the line BB' at P and Q. Prove that the quadrilateral $MPNQ$ is cyclic.

2. (Yug TST 2002) Let $ABCD$ be a quadrilateral such that $\angle A = \angle B = \angle C$. Prove that the point D, the circumcenter, and the orthocenter of $\triangle ABC$ are colinear.

3. (Republic competition 2005, 4th grade) The haxagon $ABCDEF$ is inscribed in the circle k. If the lengths of the segments AB, CD, and EF are equal to the radius of the circle k prove that the midpoints of the remaining three edges form an equilateral trinagle.

4. (USA 1997) Three isosceles triangles BCD, CAE, and ABF with the bases BC, CA, and AB respectively are constructed in the exterior of the triangle ABC. Prove that the perpendiculars from A, B, and C to the lines EF, FD, and DE repsectively are concurrent.

5. Prove that the side length of the regular 9-gon is equal to the difference of the largest and the smallest diagonal.

6. If $h_1, h_2, \dots, h_{2n}$ denote respectively the distances of an arbitrary point P of the circle k circumscribed about the polygon $A_1A_2 \dots A_{2n}$ from the lines that contain the edges A_1A_2, A_2A_3, ..., $A_{2n}A_1$, prove that $h_1h_3 \cdots h_{2n-1} = h_2h_4 \cdots h_{2n}$.

7. Let $d_1, d_2, \dots, d_n$ denote the distances of the vertices $A_1, A_2, \dots, A_n$ of the regular n-gon $A_1A_2 \dots A_n$ from an arbitrary point P of the smaller arc A_1A_n of the circumcircle. Prove that

$$\frac{1}{d_1d_2} + \frac{1}{d_2d_3} + \dots + \frac{1}{d_{n-1}d_n} = \frac{1}{d_1d_n}.$$

8. Let $A_0A_1 \dots A_{2n}$ be a regular polygon, P a point of the smaller arc A_0A_{2n} of the circumcircle and m an integer such that $0 \leq m < n$. Prove that

$$\sum_{k=0}^{n} PA_{2k}^{2m+1} = \sum_{k=1}^{n} PA_{2k-1}^{2m+1}.$$

9. (USA 2000) Let $ABCD$ be a cyclic quadrilateral and let E and F be feet of perpendiculars from the intersection of the diagonals to the lines AB and CD respectively. Prove that EF if perpendicular to the line passing through the midpoints of AD and BC.

10. Prove that the midpoints of the altitudes of the traingle are colinear if and only if the triangle is rectangular.

11. (BMO 1990) The feet of preprendiculars of the acute angled triangle ABC are A_1, B_1, and C_1. If A_2, B_2, and C_2 denote the points of tangency of the incircle of $\triangle A_1B_1C_1$ prove that the Euler lines of the triangles ABC and $A_2B_2C_2$ coincide.

12. (USA 1993) Let $ABCD$ be a convex quadrilateral whose diagonals AC and BD are perpendicular. Assume that $AC \cup BD = E$. Prove that the points symmetric to E with respect to the lines AB, BC, CD, and DA form a cyclic quadrilateral.

13. (India 1998) Let AK, BL, CM be the altitudes of the triangle ABC, and let H be its orthocenter. Let P be the midpoint of the segment AH. If BH and MK intersect at the point S, and LP and AM in the point T, prove that TS is perpendicular to BC.

14. (Vietnam 1995) Let AD, BE, and CF be the altitudes of the triangle $\triangle ABC$. For each $k \in R$, $k \neq 0$, let A_1, B_1, and C_1 be such that $AA_1 = kAD$, $BB_1 = kBE$, and $CC_1 = kCF$. Find all k such that for every non-isosceles triangle ABC the triangles ABC and $A_1B_1C_1$ are similar.

15. (Iran 2005) Let ABC be a triangle and D,E,F the points on its edges BC, CA, AB respectively such that

$$\frac{BD}{DC}=\frac{CE}{EA}=\frac{AF}{FB}=\frac{1-\lambda}{\lambda}$$

where λ is a real number. Find the locus of circumcenters of the triangles DEF as $\lambda\in\mathbf{R}$.

16. Let H_1 and H_2 be feet of perpendiculars from the orthocenter H of the triangle ABC to the bisectors of external and internal angles at the vertex C. Prove that the line H_1H_2 contains the midpoint of the side AB.

17. Given an acute-angled triangle ABC and the point D in its interior, such that $\angle ADB=\angle ACB+90^\circ$ and $AB\cdot CD=AD\cdot BC$. Find the ratio

$$\frac{AB\cdot CD}{AC\cdot BD}.$$

18. The lines AM and AN are tangent to the circle k, and an arbitrary line through A intersects k at K and L. Let l be an arbitrary line parallel to AM. Assume that KM and LM intersect the line l at P and Q, respectively. Prove that the line MN bisects the segment PQ.

19. The points D, E, and F are chosen on the edges BC, CA, and AB of the triangle ABC in such a way that $BD=CE=AF$. Prove that the triangles ABC and DEF have the common incenter if and only if ABC is equilateral.

20. Given a cyclic quadrilateral $ABCD$, prove that the incircles of the triangles ABC, BCD, CDA, DAB form an rectangle.

21. (India 1997) Let I be the incenter of the triangle ABC and let D and E be the midpoints of the segments AC and AB respectively. Assume that the lines AB and DI intersect at the point P, and the lines AC and EI at the point Q. Prove that $AP\cdot AQ=AB\cdot AC$ if and only if $\angle A=60^\circ$.

22. Let M be an interior point of the square $ABCD$. Let A_1,B_1,C_1,D_1 be the intersection of the lines AM,BM,CM,DM with the circle circumscribed about the square $ABCD$ respectively. Prove that

$$A_1B_1\cdot C_1D_1=A_1D_1\cdot B_1C_1.$$

23. Let $ABCD$ be a cyclic quadrilateral, $F=AC\cap BD$ and $E=AD\cap BC$. If M and N are the midpoints of the segments AB and CD prove that

$$\frac{MN}{EF}=\frac{1}{2}\cdot\left|\frac{AB}{CD}-\frac{CD}{AB}\right|.$$

24. (Vietnam 1994) The points A', B', and C' are symmetric to the points A, B, and C with respect to the lines BC, CA, and AB respectively. What are the conditions that $\triangle ABC$ has to satisfy in order for $\triangle A'B'C'$ to be equilateral?

25. Let O be the circumcenter of the triangle ABC and let R be its circumradius. The incircle of the triangle ABC touches the sides BC,CA,AB, at A_1,B_1,C_1 and its radius is r. Assume that the lines determined by the midpoints of AB_1 and AC_1, BA_1 and BC_1, CA_1 and CB_1 intersect at the points C_2, A_2, and B_2. Prove that the circumcenter of the triangle $A_2B_2C_2$ coincides with O, and that its circumradius is $R+\frac{r}{2}$.

26. (India 1994) Let $ABCD$ be a nonisosceles trapezoid such that $AB\parallel CD$ and $AB>CD$. Assume that $ABCD$ is circumscribed about the circle with the center I which tangets CD in E. Let M be the midpoint of the segment AB and assume that MI and CD intersect at F. Prove that $DE = FC$ if and only if $AB = 2CD$.

27. (USA 1994) Assume that the hexagon $ABCDEF$ is inscribed in the circle, $AB = CD = EF$, and that the diagonals AD, BE, and CF are concurrent. If P is the intersection of the lines AD and CE, prove that $\frac{CP}{PE} = \left(\frac{AC}{CE}\right)^2$.

28. (Vietnam 1999) Let ABC be a troiangle. The points A', B', and C' are the midpoints of the arcs BC, CA, and AB, which don't contain A, B, and C, respectively. The lines $A'B'$, $B'C'$, and $C'A'$ partition the sides of the triangle into six parts. Prove that the "middle" parts are equal if and only if the triangle ABC is equilateral.

29. (IMO 1991 shortlist) Assume that in $\triangle ABC$ we have $\angle A = 60^\circ$ and that IF is parallel to AC, where I is the incenter and F belongs to the line AB. The point P of the segment BC is such that $3BP = BC$. Prove that $\angle BFP = \angle B/2$.

30. (IMO 1997 shortlist) The angle A is the smallest in the triangle ABC. The points B and C divide the circumcircle into two arcs. Let U be the interior point of the arc between B and C which doesn't contain A. The medians of the segments AB and AC intersect the line AU respectively at the points V and W. The lines BV and CW intersect at T. Prove that $AU = TB + TC$.

31. (Vietnam 1993) Let $ABCD$ be a convex quadrilateral such that AB is not parallel to CD and AD is not parallel to BC. The points P, Q, R, and S are chosen on the edges AB, BC, CD, and DA, respectively such that $PQRS$ is a parallelogram. Find the locus of centroids of all such quadrilaterals $PQRS$.

32. The incircle of the triangle ABC touches BC, CA, AB at E, F, G respectively. Let AA_1, BB_1, CC_1 the angular bisectors of the triangle ABC (A_1, B_1, C_1 belong to the corresponding edges). Let K_A, K_B, K_C respectively be the points of tangency of the other tangents to the incircle from A_1, B_1, C_1. Let P, Q, R be the midpoints of the segments BC, CA, AB. Prove that the lines PK_A, QK_B, RK_C intersect on the incircle of the triangle ABC.

33. Assume that I and I_a are the incenter and the excenter corresponding to the edge BC of the triangle ABC. Let II_a intersect the segment BC and the circumcircle of $\triangle ABC$ at A_1 and M respectively (M belongs to I_a and I) and let N be the midpoint of the arc MBA which contains C. Assume that S and T are intersections of the lines NI and NI_a with the circumcircle of $\triangle ABC$. Prove that the points S, T, and A_1 are colinear.

34. (Vietnam 1995) Let AD, BE, CF be the altitudes of the triangle ABC, and let A', B', C' be the points on the altitudes such that

$$\frac{AA'}{AD} = \frac{BB'}{BE} = \frac{CC'}{CF} = k.$$

Find all values for k such that $\triangle A'B'C' \sim \triangle ABC$.

35. Given the triangle ABC and the point T, let P and Q be the feet of perpendiculars from T to the lines AB and AC, respectively and let R and S be the feet of perpendiculars from A to the lines TC and TB, respectively. Prove that the intersection point of the lines PR and QS belongs to the line BC.

36. (APMO 1995) Let $PQRS$ be a cyclic quadrilateral such that the lines PQ and RS are not parallel. Consider the set of all the circles passing through P and Q and all the circles passing through R and S. Determine the set of all points A of tangency of the circles from these two sets.

37. (YugMO 2003, 3-4 grade) Given a circle k and the point P outside of it. The variable line s which contains point P intersects the circle k at the points A and B. Let M and N be the midpoints of the arcs determined by the points A and B. If C is the point of the segment AB such that

$$PC^2 = PA \cdot PB,$$

prove that the measure of the angle $\angle MCN$ doesn't depend on the choice of s.

38. (YugMO 2002, 2nd grade) Let A_0,A_1,...,A_{2k}, respectevly be the points which divide the circle into $2k+1$ congruent arcs. The point A_0 is connected by the chords to all other points. Those $2k$ chords divide the circle into $2k+1$ parts. Those parts are colored alternatively in white and black in such a way that the number of white parts is by 1 bigger than the number of black parts. Prove that the surface area of teh black part is greater than the surface area of the white part.

39. (Vietnam 2003) The circles k_1 and k_2 touch each other at the point M. The radius of the circle k_1 is bigger than the radius of the circle k_2. Let A be an arbitrary point of k_2 which doesn't belong to the line connecting the centers of the circles. Let B and C be the points of k_1 such that AB and AC are its tangents. The lines BM and CM intersect k_2 again at E and F respectively. The point D is the intersection of the tangent at A with the line EF. Prove that the locus of points D (as A moves along the circle) is a line.

40. (Vietnam 2004) The circles k_1 and k_2 are given in the plane and they intersect at the points A and B. The tangents to k_1 at those points intersect at K. Let M be an arbitrary point of the circle k_1. Assume that $MA \cup k_2 = \{A,P\}$, $MK \cup k_1 = \{M,C\}$, and $CA \cup k_1 = \{A,Q\}$. Prove that the midpoint of the segment PQ belongs to the line MC and that PQ passes through a fixed point as M moves along k_1.

41. (IMO 2004 shortlist) Let $A_1A_2\dots A_n$ be a regular n-gon. Assume that the points B_1, B_2, ..., B_{n-1} are determined in the following way:

- for $i=1$ or $i=n-1$, B_i is the midpoint of the segment A_iA_{i+1};
- for $i\neq 1$, $i\neq n-1$, and S intersection of A_1A_{i+1} and A_nA_i, B_i is the intersection of the bisectors of the angle A_iS_{i+1} with A_iA_{i+1}.

Prove that $\angle A_1B_1A_n + \angle A_1B_2A_n + \ldots + \angle A_1B_{n-1}A_n = 180°$.

69. (**Dezargue's Theorem**) The triangles are perspective with respect to a point if and only if they are perspective w.r.t to a line.

42. (IMO 1998 shortlist) Let ABC be a triangle such that $\angle ACB = 2\angle ABC$. Let D be the point of the segment BC such that $CD = 2BD$. The segment AD is extended over the point D to the point E for which $AD = DE$. Prove that

$$\angle ECB + 180° = 2\angle EBC.$$

43. Given a triangle $A_1A_2A_3$ the line p passes through the point P and intersects the segments A_2A_3, A_3A_1, A_1A_2 at the points X_1, X_2, X_3, respectively. Let A_iP intersect the circumcircle of $A_1A_2A_3$ at R_i, for $i = 1,2,3$. Prove that X_1R_1, X_2R_2, X_3R_3 intersect at the point that belongs to the circumcircle of the triangle $A_1A_2A_3$.

44. The points O_1 and O_2 are the centers of the circles k_1 and k_2 that intersect. Let A be one of the intersection points of these circles. Two common tangents are constructed to these circles. BC are EF the chords of these circles with endpoints at the points of tangency of the common chords with the circles (C and F are further from A). If M and N are the midpoints of the segments BC and EF, prove that $\angle O_1AO_2 = \angle MAN = 2\angle CAF$.

45. (BMO 2002) Two circles of different radii intersect at points A and B. The common chords of these circles are MN and ST respectively. Prove that the orthocenters of $\triangle AMN$, $\triangle AST$, $\triangle BMN$, and $\triangle BST$ form a rectangle.

46. (IMO 2004 shortlist) Given a cyclic quadrilateral $ABCD$, the lines AD and BC intersect at E where C is between B and E. The diagonals AC and BD intersect at F. Let M be the midpoint of CD and let $N \neq M$ be the point of the circumcircle of the triangle ABM such that $AN/BN = AM/BM$. Prove that the points E, F, N are colinear.

47. (IMO 1994 shortlist) The diameter of the semicircle Γ belongs to the line l. Let C and D be the points on Γ. The tangents to Γ at C and D intersect the line l respectively at B and A such that the center of the semi-circle is between A and B. Let E be the intersection of the lines AC and BD, and F the foot of perpendicular from E to l. Prove that EF is the bisector of the angle $\angle CFD$.

Inversion

Dušan Djukić

Contents

1 General Properties

Inversion Ψ is a map of a plane or space without a fixed point O onto itself, determined by a circle k with center O and radius r, which takes point $A \neq O$ to the point $A' = \Psi(A)$ on the ray OA such that $OA \cdot OA' = r^2$. From now on, unless noted otherwise, X' always denotes the image of object X under a considered inversion.

Clearly, map Ψ is continuous and inverse to itself, and maps the interior and exterior of k to each other, which is why it is called "inversion". The next thing we observe is that $\triangle P'OQ' \sim \triangle QOP$ for all point $P,Q \neq O$ (for $\angle P'OQ' = \angle QOP$ and $OP'/OQ' = (r^2/OP)/(r^2/OQ) = OQ/OP$), with the ratio of similitude $\frac{r^2}{OP\cdot OQ}$. As a consequence, we have

$$\angle OQ'P' = \angle OPQ \quad \text{and} \quad P'Q' = \frac{r^2}{OP \cdot OQ} PQ.$$

What makes inversion attractive is the fact that it maps lines and circles into lines and circles. A line through O (O excluded) obviously maps to itself. What if a line p does not contain O? Let P be the projection of O on p and $Q \in p$ an arbitrary point of p. Angle $\angle OPQ = \angle OQ'P'$ is right, so Q' lies on circle k with diameter OP'. Therefore $\Psi(p) = k$ and consequently $\Psi(k) = p$. Finally, what is the image of a circle k not passing through O? We claim that it is also a circle; to show this, we shall prove that inversion takes any four concyclic points A,B,C,D to four concyclic points A',B',C',D'. The following angles are regarded as oriented. Let us show that $\angle A'C'B' = \angle A'D'B'$. We have $\angle A'C'B' = \angle OC'B' - \angle OC'A' = \angle OBC - \angle OAC$ and analogously $\angle A'D'B' = \angle OBD - \angle OAD$, which implies $\angle A'D'B' - \angle A'C'B' = \angle CBD - \angle CAD = 0$, as we claimed. To sum up:

- A line through O maps to itself.
- A circle through O maps to a line not containing O and vice-versa.
- A circle not passing through O maps to a circle not passing through O (not necessarily the same).

Remark. Based on what we have seen, it can be noted that inversion preserves angles between curves, in particular circles or lines. Maps having this property are called *conformal*.

When should inversion be used? As always, the answer comes with experience and cannot be put on a paper. Roughly speaking, inversion is useful in destroying "inconvenient" circles and angles on a picture. Thus, some pictures "cry" to be inverted:

- There are many circles and lines through the same point A. Invert through A.

Problem 1 (IMO 2003, shortlist). *Let $\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4$ be distinct circles such that Γ_1,Γ_3 are externally tangent at P, and Γ_2,Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2; Γ_2 and Γ_3; Γ_3 and Γ_4; Γ_4 and Γ_1 meet at A,B,C,D, respectively, and that all these points are different from P. Prove that*

$$\frac{AB\cdot BC}{AD\cdot DC}=\frac{PB^2}{PD^2}.$$

Solution. Apply the inversion with center at P and radius r; let $\widehat{X}$ denote the image of X. The circles $\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4$ are transformed into lines $\widehat{\Gamma_1},\widehat{\Gamma_2},\widehat{\Gamma_3},\widehat{\Gamma_4}$, where $\widehat{\Gamma_1}\parallel\widehat{\Gamma_3}$ and $\widehat{\Gamma_2}\parallel\widehat{\Gamma_4}$, and therefore $\widehat{A}\widehat{B}\widehat{C}\widehat{D}$ is a parallelogram. Further, we have $AB=\dfrac{r^2}{P\widehat{A}\cdot P\widehat{B}}\widehat{A}\widehat{B}$, $PB=\dfrac{r^2}{P\widehat{B}}$, etc. The equality to be proven becomes

$$\frac{P\widehat{D}^2}{P\widehat{B}^2}\cdot\frac{\widehat{A}\widehat{B}\cdot\widehat{B}\widehat{C}}{\widehat{A}\widehat{D}\cdot\widehat{D}\widehat{C}}=\frac{P\widehat{D}^2}{P\widehat{B}^2},$$

which holds because $\widehat{A}\widehat{B}=\widehat{C}\widehat{D}$ and $\widehat{B}\widehat{C}=\widehat{D}\widehat{A}$. $\triangle$

- There are many angles $\angle AXB$ with fixed A,B. Invert through A or B.

Problem 2 (IMO 1996, problem 2). *Let P be a point inside $\triangle ABC$ such that $\angle APB-\angle C=\angle APC-\angle B$. Let D,E be the incenters of $\triangle APB,\triangle APC$ respectively. Show that AP, BD, and CE meet in a point.*

Solution. Apply an inversion with center at A and radius r. Then the given condition becomes $\angle B'C'P'=\angle C'B'P'$, i.e., $B'P'=P'C'$. But $P'B'=\frac{r^2}{AP\cdot AB}PB$, so $AC/AB=PC/PB$. $\triangle$

Caution: Inversion may also bring new inconvenient circles and angles. Of course, keep in mind that not all circles and angles are inconvenient.

2 Problems

1. Circles k_1,k_2,k_3,k_4 are such that k_2 and k_4 each touch k_1 and k_3. Show that the tangency points are collinear or concyclic.

2. Prove that for any points A,B,C,D, $AB\cdot CD+BC\cdot DA\geq AC\cdot BD$, and that equality holds if and only if A,B,C,D are on a circle or a line in this order. (*Ptolemy's inequality*)

3. Let ω be the semicircle with diameter PQ. A circle k is tangent internally to ω and to segment PQ at C. Let AB be the tangent to k perpendicular to PQ, with A on ω and B on segment CQ. Show that AC bisects the angle $\angle PAB$.

4. Points A,B,C are given on a line in this order. Semicircles ω,ω_1,ω_2 are drawn on AC,AB,BC respectively as diameters on the same side of the line. A sequence of circles (k_n) is constructed as follows: k_0 is the circle determined by ω_2 and k_n is tangent to ω,ω_1,k_{n-1} for $n\geq 1$. Prove that the distance from the center of k_n to AB is $2n$ times the radius of k_n.

5. A circle with center O passes through points A and C and intersects the sides AB and BC of the triangle ABC at points K and N, respectively. The circumscribed circles of the triangles ABC and KBN intersect at two distinct points B and M. Prove that $\measuredangle OMB=90^\circ$. (*IMO 1985-5.*)

6. Let p be the semiperimeter of a triangle ABC. Points E and F are taken on line AB such that $CE=CF=p$. Prove that the circumcircle of $\triangle EFC$ is tangent to the excircle of $\triangle ABC$ corresponding to AB.

7. Prove that the nine-point circle of triangle ABC is tangent to the incircle and all three excircles. (*Feuerbach's theorem)*

8. The incircle of a triangle ABC is tangent to BC, CA, AB at M, N and P, respectively. Show that the circumcenter and incenter of $\triangle ABC$ and the orthocenter of $\triangle MNP$ are collinear.

9. Points A, B, C are given in this order on a line. Semicircles k and l are drawn on diameters AB and BC respectively, on the same side of the line. A circle t is tangent to k, to l at point $T \neq C$, and to the perpendicular n to AB through C. Prove that AT is tangent to l.

10. Let $A_1A_2A_3$ be a nonisosceles triangle with incenter I. Let C_i, $i = 1, 2, 3$, be the smaller circle through I tangent to A_iA_{i+1} and A_iA_{i+2} (the addition of indices being mod 3). Let B_i, $i = 1, 2, 3$, be the second point of intersection of C_{i+1} and C_{i+2}. Prove that the circumcenters of the triangles $A_1B_1I, A_2B_2I, A_3B_3I$ are collinear. (*IMO 1997 Shortlist*)

11. If seven vertices of a hexahedron lie on a sphere, then so does the eighth vertex.

12. A sphere with center on the plane of the face ABC of a tetrahedron $SABC$ passes through A, B and C, and meets the edges SA, SB, SC again at A_1, B_1, C_1, respectively. The planes through A_1, B_1, C_1 tangent to the sphere meet at a point O. Prove that O is the circumcenter of the tetrahedron $SA_1B_1C_1$.

13. Let KL and KN be the tangents from a point K to a circle k. Point M is arbitrarily taken on the extension of KN past N, and P is the second intersection point of k with the circumcircle of triangle KLM. The point Q is the foot of the perpendicular from N to ML. Prove that $\angle MPQ = 2\angle KML$.

14. The incircle Ω of the acute-angled triangle ABC is tangent to BC at K. Let AD be an altitude of triangle ABC and let M be the midpoint of AD. If N is the other common point of Ω and KM, prove that Ω and the circumcircle of triangle BCN are tangent at N. (*IMO 2002 Shortlist*)

3 Solutions

1. Let k_1 and k_2, k_2 and k_3, k_3 and k_4, k_4 and k_1 touch at A, B, C, D, respectively. An inversion with center A maps k_1 and k_2 to parallel lines k_1' and k_2', and k_3 and k_4 to circles k_3' and k_4' tangent to each other at C' and tangent to k_2' at B' and to k_4' at D'. It is easy to see that B', C', D' are collinear. Therefore B, C, D lie on a circle through A.

2. Applying the inversion with center A and radius r gives $AB = \frac{r^2}{AB'}$, $CD = \frac{r^2}{AC' \cdot AD'} C'D'$, etc. The required inequality reduces to $C'D' + B'C' \geq B'D'$.

3. Invert through C. Semicircle ω maps to the semicircle ω' with diameter $P'Q'$, circle k to the tangent to ω' parallel to $P'Q'$, and line AB to a circle l centered on $P'Q'$ which touches k (so it is congruent to the circle determined by ω'). Circle l intersects ω' and $P'Q'$ in A' and B' respectively. Hence $P'A'B'$ is an isosceles triangle with $\angle PAC = \angle A'P'C = \angle A'B'C = \angle BAC$.

4. Under the inversion with center A and squared radius $AB \cdot AC$ points B and C exchange positions, ω and ω_1 are transformed to the lines perpendicular to BC at C and B, and the sequence (k_n) to the sequence of circles (k_n') inscribed in the region between the two lines. Obviously, the distance from the center of k_n' to AB is $2n$ times its radius. Since circle k_n is homothetic to k_n' with respect to A, the statement immediately follows.

5. Invert through B. Points A', C', M' are collinear and so are K', N', M', whereas A', C', N', K' are on a circle. What does the center O of circle $ACNK$ map to? *Inversion does not preserve centers.* Let B_1 and B_2 be the feet of the tangents from B to circle $ACNK$. Their images B_1' and B_2' are the feet of the tangents from B to circle $A'C'N'K'$, and since O lies on the circle BB_1B_2,

its image O' lies on the line $B_1'B_2'$ - more precisely, it is at the midpoint of $B_1'B_2'$. We observe that M' is on the polar of point B with respect to circle $A'C'N'K'$, which is nothing but the line B_1B_2. It follows that $\angle OBM = \angle BO'M' = \angle BO'B_1' = 90^\circ$.

6. The inversion with center C and radius p maps points E and F and the excircle to themselves, and the circumcircle of $\triangle CEF$ to line AB which is tangent to the excircle. The statement follows from the fact that inversion preserves tangency.

7. We shall show that the nine-point circle ε touches the incircle k and the excircle k_a across A. Let A_1, B_1, C_1 be the midpoints of BC, CA, AB, and P, Q the points of tangency of k and k_a with BC, respectively. Recall that $A_1P = A_1Q$; this implies that the inversion with center A_1 and radius A_1P takes k and k_a to themselves. This inversion also takes ε to a line. It is not difficult to prove that this line is symmetric to BC with respect to the angle bisector of $\angle BAC$, so it also touches k and k_a.

8. The incenter of $\triangle ABC$ and the orthocenter of $\triangle MNP$ lie on the Euler line of the triangle ABC. The inversion with respect to the incircle of ABC maps points A, B, C to the midpoints of NP, PM, MN, so the circumcircle of ABC maps to the nine-point circle of $\triangle MNP$ which is also centered on the Euler line of MNP. It follows that the center of circle ABC lies on the same line.

9. An inversion with center T maps circles t and l to parallel lines t' and l', circle k and line n to circles k' and n' tangent to t' and l' (where $T \in n'$), and line AB to circle a' perpendicular to l' (because an inversion preserves angles) and passes through $B', C' \in l'$; thus a' is the circle with diameter $B'C'$. Circles k' and n' are congruent and tangent to l' at B' and C', and intersect a' at A' and T respectively. It follows that A' and T are symmetric with respect to the perpendicular bisector of $B'C'$ and hence $A'T \parallel l'$, so AT is tangent to l.

10. The centers of three circles passing through the same point I and not touching each other are collinear if and only if they have another common point. Hence it is enough to show that the circles A_iB_iI have a common point other than I. Now apply inversion at center I and with an arbitrary power. We shall denote by X' the image of X under this inversion. In our case, the image of the circle C_i is the line $B_{i+1}'B_{i+2}'$ while the image of the line $A_{i+1}A_{i+2}$ is the circle $IA_{i+1}'A_{i+2}'$ that is tangent to $B_i'B_{i+2}'$, and $B_i'B_{i+2}'$. These three circles have equal radii, so their centers P_1, P_2, P_3 form a triangle also homothetic to $\triangle B_1'B_2'B_3'$. Consequently, points A_1', A_2', A_3', that are the reflections of I across the sides of $P_1P_2P_3$, are vertices of a triangle also homothetic to $B_1'B_2'B_3'$. It follows that $A_1'B_1', A_2'B_2', A_3'B_3'$ are concurrent at some point J', i.e., that the circles A_iB_iI all pass through J.

11. Let $AYBZ, AZCX, AXDY, WCXD, WDYB, WBZC$ be the faces of the hexahedron, where A is the "eighth" vertex. Apply an inversion with center W. Points B', C', D', X', Y', Z' lie on some plane π, and moreover, C', X', D'; D', Y', B'; and B', Z', C' are collinear in these orders. Since A is the intersection of the planes YBZ, ZCX, XDY, point A' is the second intersection point of the spheres $WY'B'Z', WZ'C'X', WX'D'Y'$. Since the circles $Y'B'Z', Z'C'X', X'D'Y'$ themselves meet at a point on plane π, this point must coincide with A'. Thus $A' \in \pi$ and the statement follows.

12. Apply the inversion with center S and squared radius $SA \cdot SA_1 = SB \cdot SB_1 = SC \cdot SC_1$. Points A and A_1, B and B_1, and C and C_1 map to each other, the sphere through A, B, C, A_1, B_1, C_1 maps to itself, and the tangent planes at A_1, B_1, C_1 go to the spheres through S and A, S and B, S and C which touch the sphere $ABCA_1B_1C_1$. These three spheres are perpendicular to the plane ABC, so their centers lie on the plane ABC; hence they all pass through the point $\overline{S}$ symmetric to S with respect to plane ABC. Therefore $\overline{S}$ is the image of O. Now since $\angle SA_1O = \angle S\overline{S}A = \angle \overline{S}SA = \angle OSA_1$, we have $OS = OA_1$ and analogously $OS = OB_1 = OC_1$.

13. Apply the inversion with center M. Line MN' is tangent to circle k' with center O', and a circle through M is tangent to k' at L' and meets MN' again at K'. The line $K'L'$ intersects k' at P', and $N'O'$ intersects ML' at Q'. The task is to show that $\angle MQ'P' = \angle L'Q'P' = 2\angle K'ML'$.

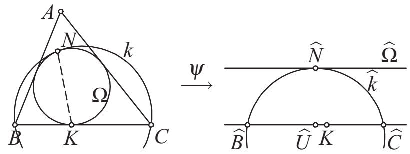
    Let the common tangent at L' intersect MN' at Y'. Since the peripheral angles on the chords $K'L'$ and $L'P'$ are equal (to $\angle K'L'Y'$), we have $\angle L'O'P' = 2\angle L'N'P' = 2\angle K'ML'$. It only remains to show that L', P', O', Q' are on a circle. This follows from the equality $\angle O'Q'L' = 90^\circ - \angle L'MK' = 90^\circ - \angle L'N'P' = \angle O'P'L'$ (the angles are regarded as oriented).

14. Let k be the circle through B, C that is tangent to the circle Ω at point N'. We must prove that K, M, N' are collinear. Since the statement is trivial for $AB = AC$, we may assume that $AC > AB$. As usual, $R, r, \alpha, \beta, \gamma$ denote the circumradius and the inradius and the angles of $\triangle ABC$, respectively.

    We have $\tan\angle BKM = DM/DK$. Straightforward calculation gives $DM = \frac{1}{2}AD = R\sin\beta\sin\gamma$ and $DK = \frac{DC-DB}{2} - \frac{KC-KB}{2} = R\sin(\beta-\gamma) - R(\sin\beta - \sin\gamma) = 4R\sin\frac{\beta-\gamma}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$, so we obtain

    $$\tan\angle BKM = \frac{\sin\beta\sin\gamma}{4\sin\frac{\beta-\gamma}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}} = \frac{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\sin\frac{\beta-\gamma}{2}}.$$

    To calculate the angle BKN', we apply the inversion ψ with center at K and power $BK \cdot CK$. For each object X, we denote by $\widehat{X}$ its image under ψ. The incircle Ω maps to a



    line $\widehat{\Omega}$ parallel to $\widehat{B}\widehat{C}$, at distance $\frac{BK \cdot CK}{2r}$ from $\widehat{B}\widehat{C}$. Thus the point $\widehat{N'}$ is the projection of the midpoint $\widehat{U}$ of $\widehat{B}\widehat{C}$ onto $\widehat{\Omega}$. Hence

    $$\tan\angle BKN' = \tan\angle \widehat{B}K\widehat{N'} = \frac{\widehat{U}\widehat{N'}}{\widehat{U}K} = \frac{BK \cdot CK}{r(CK - BK)}.$$

    Again, one easily checks that $KB \cdot KC = bc\sin^2\frac{\alpha}{2}$ and $r = 4R\sin\frac{\alpha}{2} \cdot \sin\frac{\beta}{2} \cdot \sin\frac{\gamma}{2}$, which implies

    $$\begin{aligned} \tan\angle BKN' &= \frac{bc\sin^2\frac{\alpha}{2}}{r(b-c)} \\ &= \frac{4R^2\sin\beta\sin\gamma\sin^2\frac{\alpha}{2}}{4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \cdot 2R(\sin\beta - \sin\gamma)} = \frac{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\sin\frac{\beta-\gamma}{2}}. \end{aligned}$$

    Hence $\angle BKM = \angle BKN'$, which implies that K, M, N' are indeed collinear; thus $N' \equiv N$.

Projective Geometry

Milivoje Lukić

Contents

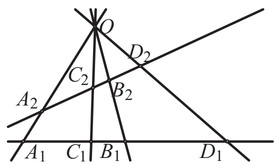
1 Cross Ratio. Harmonic Conjugates. Perspectivity. Projectivity

Definition 1. *Let A, B, C, and D be colinear points. The* cross ratio *of the pairs of points (A,B) and (C,D) is*

$$\mathscr{R}(A,B;C,D) = \frac{\overrightarrow{AC}}{\overrightarrow{CB}} : \frac{\overrightarrow{AD}}{\overrightarrow{DB}}. \tag{1}$$

Let a, b, c, d be four concurrent lines. For the given lines p_1 and p_2 let us denote $A_i = a \cap p_i$, $B_i = b \cap p_i$, $C_i = c \cap p_i$, $D_i = d \cap p_i$, for $i = 1, 2$. Then

$$\mathscr{R}(A_1,B_1;C_1,D_1) = \mathscr{R}(A_2,B_2;C_2,D_2). \tag{2}$$

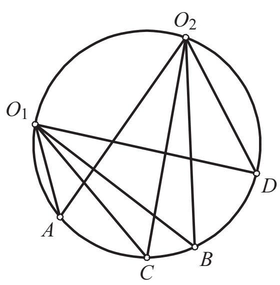


Thus it is meaningful to define the cross ratio of the pairs of concurrent points as

$$\mathscr{R}(a,b;c,d) = \mathscr{R}(A_1,B_1;C_1,D_1). \tag{3}$$

Assume that points O_1, O_2, A, B, C, D belong to a circle. Then

$$\begin{aligned} &\mathscr{R}(O_1A,O_1B;O_1C,O_1D) \\ = \;&\mathscr{R}(O_2A,O_2B;O_2C,O_2D). \end{aligned} \tag{4}$$

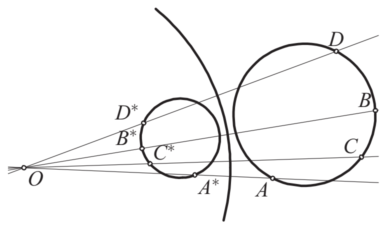


Hence it is meaningful to define the cross-ratio for cocyclic points as

$$\mathscr{R}(A,B;C,D) = \mathscr{R}(O_1A,O_1B;O_1C,O_1D). \tag{5}$$

Assume that the points A, B, C, D are colinear or cocyclic. Let an inversion with center O maps A, B, C, D into A^*, B^*, C^*, D^*. Then

$$\mathscr{R}(A,B;C,D) = \mathscr{R}(A^*,B^*;C^*,D^*). \qquad (6)$$



Definition 2. *Assume that A, B, C, and D are cocyclic or colinear points. Pairs of points (A,B) and (C,D) are* harmonic conjugates *if $\mathscr{R}(A,B;C,D) = -1$. We also write $\mathscr{H}(A,B;C,D)$ when we want to say that (A,B) and (C,D) are harmonic conjugates to each other.*

Definition 3. *Let each of l_1 and l_2 be either line or circle.* Perspectivity *with respect to the point S $\overset{S}{\barwedge}$, is the mapping of $l_1 \to l_2$, such that*

(i) If either l_1 or l_2 is a circle than it contains S;

(ii) every point $A_1 \in l_1$ is mapped to the point $A_2 = OA_1 \cap l_2$.

According to the previous statements perspectivity preserves the cross ratio and hence the harmonic conjugates.

Definition 4. *Let each of l_1 and l_2 be either line or circle.* Projectivity *is any mapping from l_1 to l_2 that can be represented as a finite composition of perspectivities.*

Theorem 1. *Assume that the points A, B, C, D_1, and D_2 are either colinear or cocyclic. If the equation $\mathscr{R}(A,B;C,D_1) = \mathscr{R}(A,B;C,D_2)$ is satisfied, then $D_1 = D_2$. In other words, a projectivity with three fixed points is the identity.*

Theorem 2. *If the points A, B, C, D are mutually discjoint and $\mathscr{R}(A,B;C,D) = \mathscr{R}(B,A;C,D)$ then $\mathscr{H}(A,B;C,D)$.*

2 Desargue's Theorem

The triangles $A_1B_1C_1$ and $A_2B_2C_2$ are *perspective with respect to a center* if the lines A_1A_2, B_1B_2, and C_1C_2 are concurrent. They are *perspective with respect to an axis* if the points $K = B_1C_1 \cap B_2C_2$, $L = A_1C_1 \cap A_2C_2$, $M = A_1B_1 \cap A_2B_2$ are colinear.

Theorem 3 (Desargue). *Two triangles are perspective with respect to a center if and only if they are perspective with respect to a point.*

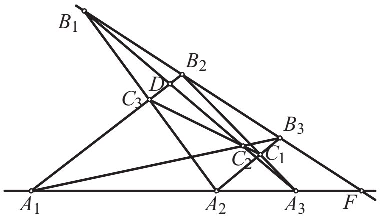
3 Theorems of Pappus and Pascal

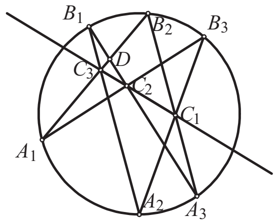
Theorem 4 (Pappus). *The points A_1, A_2, A_3 belong to the line a, and the points B_1, B_2, B_3 belong to the line b. Assume that $A_1B_2 \cap A_2B_1 = C_3$, $A_1B_3 \cap A_3B_1 = C_2$, $A_2B_3 \cap A_3B_2 = C_1$. Then C_1, C_2, C_3 are colinear.*

Proof. Denote $C_2' = C_1C_3 \cap A_3B_1$, $D = A_1B_2 \cap A_3B_1$, $E = A_2B_1 \cap A_3B_2$, $F = a \cap b$. Our goal is to prove that the points C_2 and C_2' are identical. Consider the sequence of projectivities:

$$A_3B_1DC_2 \overset{A_1}{\barwedge} FB_1B_2B_3 \overset{A_2}{\barwedge} A_3EB_2C_1 \overset{C_3}{\barwedge} A_3B_1DC_2'.$$

We have got the projective transformation of the line A_3B_1 that fixes the points A_3, B_1, D, and maps C_2 to C_2'. Since the projective mapping with three fixed points is the identity we have $C_2 = C_2'$. □





Theorem 5 (Pascal). *Assume that the points A_1, A_2, A_3, B_1, B_2, B_3 belong to a circle. The point in intersections of A_1B_2 with A_2B_1, A_1B_3 with A_3B_1, A_2B_3 with A_3B_2 lie on a line.*

Proof. The points C_2', D, and E as in the proof of the Pappus theorem. Consider the sequence of perspectivities

$$A_3B_1DC_2 \overset{A_1}{\overline{\wedge}} A_3B_1B_2B_3 \overset{A_2}{\overline{\wedge}} A_3EB_2C_1 \overset{C_3}{\overline{\wedge}} A_3B_1DC_2'.$$

In the same way as above we conclude that $C_2 = C_2'$. □

4 Pole. Polar. Theorems of Brianchon and Brokard

Definition 5. *Given a circle $k(O,r)$, let A^* be the image of the point $A \neq O$ under the inversion with respect to k. The line a passing through A^* and perpendicular to OA is called the* polar *of A with respect to k. Conversely A is called the* pole *of a with respect to k.*

Theorem 6. *Given a circle $k(O,r)$, let and a and b be the polars of A and B with respect to k. The $A \in b$ if and only if $B \in a$.*

Proof. $A \in b$ if and only if $\angle AB^*O = 90^\circ$. Analogously $B \in a$ if and only if $\angle BA^*O = 90^\circ$, and it reamins to notice that according to the basic properties of inversion we have $\angle AB^*O = \angle BA^*O$. □

Definition 6. *Points A and B are called* conjugated *with respect to the circle k if one of them lies on a polar of the other.*

Theorem 7. *If the line determined by two conjugated points A and B intersects $k(O,r)$ at C and D, then $\mathscr{H}(A,B;C,D)$. Conversely if $\mathscr{H}(A,B;C,D)$, where $C,D \in k$ then A and B are conjugated with respect to k.*

Proof. Let C_1 and D_1 be the intersection points of OA with k. Since the inversion preserves the cross-ratio and $\mathscr{R}(C_1,D_1;A,A^*) = \mathscr{R}(C_1,D_1;A^*,A)$ we have

$$\mathscr{H}(C_1,D_1;A,A^*). \tag{7}$$

Let p be the line that contains A and intersects k at C and D. Let $E = CC_1 \cap DD_1$, $F = CD_1 \cap DC_1$. Since C_1D_1 is the diameter of k we have $C_1F \perp D_1E$ and $D_1F \perp C_1E$, hence F is the orthocenter of the triangle C_1D_1E. Let $B = EF \cap CD$ and $\overline{A}^* = EF \cap C_1D_1$. Since

$$C_1D_1A\overline{A}^* \overset{E}{\overline{\wedge}} CDAB \overset{F}{\overline{\wedge}} D_1C_1A\overline{A}^*$$

have $\mathscr{H}(C_1,D_1;A,\overline{A}^*)$ and $\mathscr{H}(C,D;A,B)$. (7) now implies two facts:

1° From $\mathscr{H}(C_1,D_1;A,\overline{A}^*)$ and $\mathscr{H}(C_1,D_1;A,A^*)$ we get $A^* = \overline{A}^*$, hence $A^* \in EF$. However, since $EF \perp C_1D_1$, the line $EF = a$ is the polar of A.

2° For the point B which belongs to the polar of A we have $\mathscr{H}(C,D;A,B)$. This completes the proof. □

Theorem 8 (Brianchon's theorem). *Assume that the hexagon $A_1A_2A_3A_4A_5A_6$ is circumscribed about the circle k. The lines A_1A_4, A_2A_5, and A_3A_6 intersect at a point.*

Proof. We will use the convention in which the points will be denoted by capital latin letters, and their repsective polars with the corresponding lowercase letters.

Denote by M_i, $i = 1,2,\dots,6$, the points of tangency of A_iA_{i+1} with k. Since $m_i = A_iA_{i+1}$, we have $M_i \in a_i$, $M_i \in a_{i+1}$, hence $a_i = M_{i-1}M_i$.

Let $b_j = A_jA_{j+3}$, $j = 1,2,3$. Then $B_j = a_j \cap a_{j+3} = M_{j-1}M_j \cap M_{j+3}M_{j+4}$. We have to prove that there exists a point P such that $P \in b_1, b_2, b_3$, or analogously, that there is a line p such that $B_1, B_2, B_3 \in p$. In other words we have to prove that the points B_1, B_2, B_3 are colinear. However this immediately follows from the Pascal's theorem applied to $M_1M_3M_5M_4M_6M_2$. $\square$

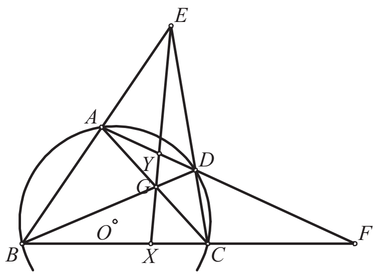
From the previous proof we see that the Brianchon's theorem is obtained from the Pascal's by replacing all the points with their polars and all lines by theirs poles.

Theorem 9 (Brokard). *The quadrilateral $ABCD$ is inscribed in the circle k with center O. Let $E = AB \cap CD$, $F = AD \cap BC$, $G = AC \cap BD$. Then O is the orthocenter of the triangle EFG.*

Proof. We will prove that EG is a polar of F. Let $X = EG \cap BC$ and $Y = EG \cap AD$. Then we also have

$$ADYF \overset{E}{\barwedge} BCXF \overset{G}{\barwedge} DAYF,$$

which implies the relations $\mathscr{H}(A,D;Y,F)$ and $\mathscr{H}(B,C;X,F)$. According to the properties of polar we have that the points X and Y lie on a polar of the point F, hence EG is a polar of the point F.



Since EG is a polar of F, we have $EG \perp OF$. Analogously we have $FG \perp OE$, thus O is the orthocenter of $\triangle EFG$. $\square$

5 Problems

1. Given a quadrilateral $ABCD$, let $P = AB \cap CD$, $Q = AD \cap BC$, $R = AC \cap PQ$, $S = BD \cap PQ$. Prove that $\mathscr{H}(P,Q;R,S)$.

2. Given a triangle ABC and a point M on BC, let N be the point of the line BC such that $\angle MAN = 90^\circ$. Prove that $\mathscr{H}(B,C;M,N)$ if and only if AM is the bisector of the angle $\angle BAC$.

3. Let A and B be two points and let C be the point of the line AB. Using just a ruler find a point D on the line AB such that $\mathscr{H}(A,B;C,D)$.

4. Let A, B, C be the diagonal points of the quadrilateral $PQRS$, or equivalently $A = PQ \cap RS$, $B = QR \cap SP$, $C = PR \cap QS$. If only the points A, B, C, S, are given using just a ruler construct the points P, Q, R.

5. Assume that the incircle of $\triangle ABC$ touches the sides BC, AC, and AB at D, E, and F. Let M be the point such that the circle k_1 incscibed in $\triangle BCM$ touches BC at D, and the sides BM and CM at P and Q. Prove that the lines EF, PQ, BC are concurrent.

6. Given a triangle ABC, let D and E be the points on BC such that $BD = DE = EC$. The line p intersects AB, AD, AE, AC at K, L, M, N, respectively. Prove that $KN \geq 3LM$.

7. The point M_1 belongs to the side AB of the quadrilateral $ABCD$. Let M_2 be the projection of M_1 to the line BC from D, M_3 projection of M_2 to CD from A, M_4 projection of M_3 to DA from B, M_5 projection of M_4 to AB from C, etc. Prove that $M_{13} = M_1$.

8. (butterfly theorem) Points M and N belong to the circle k. Let P be the midpoint of the chord MN, and let AB and CD (A and C are on the same side of MN) be arbitrary chords of k passing through P. Prove that lines AD and BC intersect MN at points that are equidistant from P.

9. Given a triangle ABC, let D and E be the points of the sides AB and AC respectively such that $DE \| BC$. Let P be an interior point of the triangle ADE. Assume that the lines BP and CP intersect DE at F and G respectively. The circumcircles of $\triangle PDG$ and $\triangle PFE$ intersect at P and Q. Prove that the points A, P, and Q are colinear.

10. (IMO 1997 shortlist) Let $A_1A_2A_3$ be a non-isosceles triangle with the incenter I. Let C_i, $i = 1$, 2, 3, be the smaller circle through I tangent to both A_iA_{i+1} and A_iA_{i+2} (summation of indeces is done modulus 3). Let B_i, $i = 1$, 2, 3, be the other intersection point of C_{i+1} and C_{i+2}. Prove that the circumcenters of the triangles A_1B_1I, A_2B_2I, A_3B_3I are colinear.

11. Given a triangle ABC and a point T, let P and Q be the feet of perpendiculars from T to the lines AB and AC, respectively. Let R and S be the feet of perpendiculars from A to TC and TB, respectively. Prove that the intersection of PR and QS belongs to BC.

12. Given a triangle ABC and a point M, a line passing through M intersects AB, BC, and CA at C_1, A_1, and B_1, respectively. The lines AM, BM, and CM intersect the circumcircle of $\triangle ABC$ repsectively at A_2, B_2, and C_2. Prove that the lines A_1A_2, B_1B_2, and C_1C_2 intersect in a point that belongs to the circumcircle of $\triangle ABC$.

13. Let P and Q isogonaly conjugated points and assume that $\triangle P_1P_2P_3$ and $\triangle Q_1Q_2Q_3$ are their pedal triangles, respectively. Let $X_1 = P_2Q_3 \cap P_3Q_2$, $X_2 = P_1Q_3 \cap P_3Q_1$, $X_3 = P_1Q_2 \cap P_2Q_1$. Prove that the points X_1, X_2, X_3 belong to the line PQ.

14. If the points A and M are conjugated with respect to k, then the circle with diameter AM is orthogonal to k.

15. From a point A in the exterior of a circle k two tangents AM and AN are drawn. Assume that K and L are two points of k such that A, K, L are colinear. Prove that MN bisects the segment PQ.

16. The point isogonaly conjugated to the centroid is called the *Lemuan* point. The lines connected the vertices with the Lemuan point are called *symmedians*. Assume that the tangents from B and C to the circumcircle Γ of $\triangle ABC$ intersect at the point P. Prove that AP is a symmedian of $\triangle ABC$.

17. Given a triangle ABC, assume that the incircle touches the sides BC, CA, AB at the points M, N, P, respectively. Prove that AM, BN, and CP intersect in a point.

18. Let $ABCD$ be a quadrilateral circumscribed about a circle. Let M, N, P, and Q be the points of tangency of the incircle with the sides AB, BC, CD, and DA respectively. Prove that the lines AC, BD, MP, and NQ intersect in a point.

19. Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD intersect at O; extensions of the sides AB and CD at E; the tangents to the circumcircle from A and D at K; and the tangents to the circumcircle at B and C at L. Prove that the points E, K, O, and L lie on a line.

20. Let $ABCD$ be a cyclic quadrilateral. The lines AB and CD intersect at the point E, and the diagonals AC and BD at the point F. The circumcircle of the triangles $\triangle AFD$ and $\triangle BFC$ intersect again at H. Prove that $\angle EHF = 90^\circ$.

6 Solutions

1. Let $T = AC \cap BD$. Consider the sequence of the perspectivities
$$PQRS \overset{A}{\barwedge} BDTS \overset{C}{\barwedge} QPRS.$$
Since the perspectivity preserves the cross-ratio $\mathscr{R}(P,Q;R,S) = \mathscr{R}(Q,P;R,S)$ we obtain that $\mathscr{H}(P,Q;R,S)$.

2. Let $\alpha = \angle BAC$, $\beta = \angle CBA$, $\gamma = \angle ACB$ and $\varphi = \angle BAM$. Using the sine theorem on $\triangle ABM$ and $\triangle ACM$ we get
$$\frac{BM}{MC} = \frac{BM}{AM}\frac{AM}{CM} = \frac{\sin\varphi}{\sin\beta}\,\frac{\sin\gamma}{\sin(\alpha-\varphi)}.$$
Similarly using the sine theorem on $\triangle ABN$ and $\triangle ACN$ we get
$$\frac{BN}{NC} = \frac{BN}{AN}\frac{AN}{CN} = \frac{\sin(90^\circ-\varphi)}{\sin(180^\circ-\beta)}\,\frac{\sin\gamma}{\sin(90^\circ+\alpha-\varphi)}.$$
Combining the previous two equations we get
$$\frac{BM}{MC} : \frac{BN}{NC} = \frac{\tan\varphi}{\tan(\alpha-\varphi)}.$$
Hence, $|\mathscr{R}(B,C;M,N)| = 1$ is equivalent to $\tan\varphi = \tan(\alpha-\varphi)$, i.e. to $\varphi = \alpha/2$. Since $B \neq C$ and $M \neq N$, the relation $|\mathscr{R}(B,C;M,N)| = 1$ is equivalent to $\mathscr{R}(B,C;M,N) = -1$, and the statement is now shown.

3. The motivation is the problem 1. Choose a point K outside AB and point L on AK different from A and K. Let $M = BL \cap CK$ and $N = BK \cap AM$. Now let us construct a point D as $D = AB \cap LN$. From the problem 1 we indeed have $\mathscr{H}(A,B;C,D)$.

4. Let us denote $D = AS \cap BC$. According to the problem 1 we have $\mathscr{H}(R,S;A,D)$. Now we construct the point $D = AS \cap BC$. We have the points A, D, and S, hence according to the previous problem we can construct a point R such that $\mathscr{H}(A,D;S,R)$. Now we construct $P = BS \cap CR$ and $Q = CS \cap BR$, which solves the problem.

5. It is well known (and is easy to prove using Ceva's theorem) that the lines AD, BE, and CF intersect at a point G (called a Gergonne point of $\triangle ABC$) Let $X = BC \cap EF$. As in the problem 1 we have $\mathscr{H}(B,C;D,X)$. If we denote $X' = BC \cap PQ$ we analogously have $\mathscr{H}(B,C;D,X')$, hence $X = X'$.

6. Let us denote $x = KL$, $y = LM$, $z = MN$. We have to prove that $x+y+z \geq 3y$, or equivalently $x+z \geq 2y$. Since $\mathscr{R}(K,N;L,M) = \mathscr{R}(B,C;D,E)$, we have
$$\frac{x}{y+z} : \frac{x+y}{z} = \frac{\overrightarrow{KL}}{\overrightarrow{LN}} : \frac{\overrightarrow{KM}}{\overrightarrow{MN}} = \frac{\overrightarrow{BD}}{\overrightarrow{DC}} : \frac{\overrightarrow{BE}}{\overrightarrow{EC}} = \frac{1}{2} : \frac{1}{2},$$
implying $4xz = (x+y)(y+z)$.

If it were $y > (x+z)/2$ we would have
$$x+y > \frac{3}{2}x + \frac{1}{2}z = 2\frac{1}{4}(x+x+x+z) \geq 2\sqrt[4]{xxxz},$$
and analogously $y+z > 2\sqrt[4]{xzzz}$ as well as $(x+y)(y+z) > 4xz$ which is a contradiction. Hence the assumption $y > (x+z)/2$ was false so we have $y \leq (x+z)/2$.

Let us analyze the case of equality. If $y = (x+z)/2$, then $4xz = (x+y)(x+z) = (3x+z)(x+3z)/4$, which is equivalent to $(x-z)^2 = 0$. Hence the equality holds if $x = y = z$. We leave to the reader to prove that $x = y = z$ is satisfied if and only if $p \parallel BC$.

7. Let $E = AB \cap CD$, $F = AD \cap BC$. Consider the sequence of perspectivities
$$ABEM_1 \overset{D}{\barwedge} FBCM_2 \overset{A}{\barwedge} DECM_3 \overset{B}{\barwedge} DAFM_4 \overset{C}{\barwedge} EABM_5. \qquad (8)$$
According to the conditions given in the problem this sequence of perspectivites has two be applied three more times to arrive to the point M_{13}. Notice that the given sequence of perspectivities maps A to E, E to B, and B to A. Clearly if we apply (8) three times the points A, B, and E will be fixed while M_1 will be mapped to M_{13}. Thus $M_1 = M_{13}$.

8. Let X' be the point symmetric to Y with respect to P. Notice that
$$\begin{aligned}\mathscr{R}(M,N;X,P) &= \mathscr{R}(M,N;P,Y) \quad (\text{from } MNXP \overset{D}{\barwedge} MNAC \overset{B}{\barwedge} MNPY)\\ &= \mathscr{R}(N,M;P,X') \quad (\text{ the reflection with the center } P \text{ preserves}\\ &\qquad\qquad\qquad\qquad \text{the ratio, hence it preserves the cross-ratio})\\ &= \frac{1}{\mathscr{R}(N,M;X',P)} = \mathscr{R}(M,N;X',P),\end{aligned}$$
where the last equality follows from the basic properties of the cross ratio. It follows that $X = X'$.

9. Let $J = DQ \cap BP$, $K = EQ \cap CP$. If we prove that $JK \parallel DE$ this would imply that the triangles BDJ and CEK are perspective with the respect to a center, hence with repsect to an axis as well (according to Desargue's theorem) which immediately implies that A, P, Q are colinear (we encourage the reader to verify this fact).

   Now we will prove that $JK \| DE$. Let us denote $T = DE \cap PQ$. Applying the Menelaus theorem on the triangle DTQ and the line PF we get
$$\frac{\overrightarrow{DJ}}{\overrightarrow{JQ}}\frac{\overrightarrow{QP}}{\overrightarrow{PT}}\frac{\overrightarrow{TF}}{\overrightarrow{FD}} = -1.$$
   Similarly from the triangle ETQ and the line PG:
$$\frac{\overrightarrow{EK}}{\overrightarrow{KQ}}\frac{\overrightarrow{QP}}{\overrightarrow{PT}}\frac{\overrightarrow{TG}}{\overrightarrow{GE}} = -1.$$
   Dividing the last two equalities and using $DT \cdot TG = FT \cdot TE$ (T is on the radical axis of the circumcircles of $\triangle DPG$ and $\triangle FPE$), we get
$$\frac{\overrightarrow{DJ}}{\overrightarrow{JQ}} = \frac{\overrightarrow{EK}}{\overrightarrow{KQ}}.$$
   Thus $JK \parallel DE$, q.e.d.

10. Apply the inversion with the respect to I. We leave to the reader to draw the inverse picture. Notice that the condition that I is the incentar now reads that the circumcircles $A_i^* A_{i+1}^* I$ are of the same radii. Indeed if R is the radius of the circle of inversion and r the distance between I and XY then the radius of the circumcircle of $\triangle IX^*Y^*$ is equal to R^2/r. Now we use the following statement that is very easy to prove: "Let k_1, k_2, k_3 be three circles such that all pass through the same point I, but no two of them are mutually tangent. Then the centers of these circles are colinear if and only if there exists another common point $J \neq I$ of these three circles."

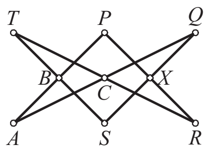
    In the inverse picture this transforms into proving that the lines $A_1^*B_1^*$, $A_2^*B_2^*$, and $A_3^*B_3^*$ intersect at a point.

In order to prove this it is enough to show that the corresponding sides of the triangles $A_1^*A_2^*A_3^*$ and $B_1^*B_2^*B_3^*$ are parallel (then these triangles would be perspective with respect to the infinitely far line). Afterwards the Desargue's theorem would imply that the triangles are perspective with respect to a center. Let P_i^* be the incenter of $A_{i+1}^*A_{i+2}^*I$, and let Q_i^* be the foot of the perpendicular from I to $P_{i+1}^*P_{i+2}^*$. It is easy to prove that

$$\overrightarrow{A_1^*A_2^*} = 2\overrightarrow{Q_1^*Q_2^*} = -\overrightarrow{P_1^*P_2^*}.$$

Also since the circles $A_i^*A_{i+1}^*I$ are of the same radii, we have $P_1^*P_2^* \parallel B_1^*B_2^*$, hence $A_1^*A_2^* \parallel B_1^*B_2^*$.

11. We will prove that the intersection X of PR and QS lies on the line BC. Notice that the points P, Q, R, S belong to the circle with center AT. Consider the six points A, S, R, T, P, Q that lie on a circle. Using Pascal's theorem with respect to the diagram



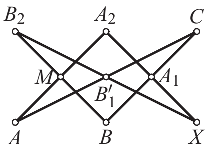
we get that the points B, C, and $X = PR \cap QS$ are colinear.

12. *First solution, using projective mappings.* Let $A_3 = AM \cap BC$ and $B_3 = BM \cap AC$. Let X be the other intersection point of the line A_1A_2 with the circumcircle k of $\triangle ABC$. Let X' be the other intersection point of the line B_1B_2 with k. Consider the sequence of perspectivities

$$ABCX \overset{A_2}{\barwedge} A_3BCA_1 \overset{M}{\barwedge} AB_3CB_1 \overset{B_2}{\barwedge} ABCX'$$

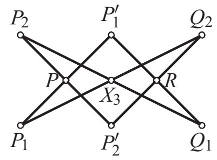
which has three fixed points A, B, C, hence $X = X'$. Analogously the line C_1C_2 contains X and the problem is completely solved.

Second solution, using Pascal's theorem. Assume that the line A_1A_2 intersect the circumcircle of the trianlge ABC at A_2 and X. Let $XB_2 \cap AC = B_1'$. Let us apply the Pascal's theorem on the points A, B, C, A_2, B_2, X according the diagram:



It follows that the points A_1, B_1', and M are colinear. Hence $B_1' \in A_1M$. According to the definition of the point B_1' we have $B_1' \in AC$ hence $B_1' = A_1M \cap AC = B_1$. The conclusion is that the points X, B_1, B_2 are colinear. Analogously we prove that the points X, C_1, C_2 are colinear, hence the lines A_1A_2, B_1B_2, C_1C_2 intersect at X that belongs to the circumcircle of the triangle ABC.

13. It is well known (from the theory of pedal triangles) that pedal triangles corresponding to the isogonally conjugated points have the common circumcircle, so called *pedal circle* of the points P and Q. The center of that circle which is at the same time the midpoint of PQ will be denoted by R. Let $P_1' = PP_1 \cap Q_1R$ and $P_2' = PP_2 \cap Q_2R$ (the points P_1' and P_2' belong to the pedal circle of the point P, as point on the same diameters as Q_1 and Q_2 respectively). Using the Pascal's theorem on the points Q_1, P_2, P_2', Q_2, P_1, P_1' in the order shown by the diagram



we get that the points P, R, X_1 are colinear or $X_1 \in PQ$. Analogously the points X_2, X_3 belong to the line PQ.

14. Let us recall the statement according to which the circle l is invariant under the inversion with respect to the circle k if and only if $l = k$ or $l \perp k$.

    Since the point M belongs to the polar of the point A with respect to k we have $\angle MA^*A = 90^\circ$ where $A^* = \psi_l(A)$. Therefore $A^* \in l$ where l is the circle with the radius AM. Analogously $M^* \in l$. However from $A \in l$ we get $A^* \in l^*$; $A^* \in l$ yields $A \in l^*$ (the inversion is inverse to itself) hence $\psi_l(A^*) = A$). Similarly we get $M \in l^*$ and $M^* \in l^*$. Notice that the circles l and l^* have the four common points A, A^*, M, M^*, which is exactly two too much. Hence $l = l^*$ and according to the statement mentioned at the beginning we conclude $l = k$ or $l \perp k$. The case $l = k$ can be easily eliminated, because the circle l has the diameter AM, and AM can't be the diameter of k because A and M are conjugated to each other.

    Thus $l \perp k$, q.e.d.

15. Let $J = KL \cap MN$, $R = l \cap MN$, $X_\infty = l \cap AM$. Since MN is the polar of A from $J \in MN$ we get $\mathscr{H}(K,L;J,A)$. From $KLJA \overset{M}{\barwedge} PQRX_\infty$ we also have $\mathscr{H}(P,Q;R,X_\infty)$. This implies that R is the midpoint of PQ.

16. Let Q be the intersection point of the lines AP and BC. Let Q' be the point of BC such that the ray AQ' is isogonal to the ray AQ in the triangle ABC. This exactly means that $\angle Q'AC = \angle BAQ$ i $\angle BAQ' = \angle QAC$.

    For an arbitrary point X of the segment BC, the sine theorem applied to triangles BAX and XAC yields

    $$\frac{BX}{XC} = \frac{BX}{AX}\frac{AX}{XC} = \frac{\sin\angle BAX}{\sin\angle ABX}\frac{\sin\angle ACX}{\sin\angle XAC} = \frac{\sin\angle ACX}{\sin\angle ABX}\frac{\sin\angle BAX}{\sin\angle XAC} = \frac{AB}{AC}\frac{\sin\angle BAX}{\sin\angle XAC}.$$

    Applying this to $X = Q$ and $X = Q'$ and multiplying together afterwards we get

    $$\frac{BQ}{QC}\frac{BQ'}{Q'C} = \frac{AB}{AC}\frac{\sin\angle BAQ}{\sin\angle QAC}\frac{AB}{AC}\frac{\sin\angle BAQ'}{\sin\angle Q'AC} = \frac{AB^2}{AC^2}. \tag{9}$$

    Hence if we prove $BQ/QC = AB^2/AC^2$ we would immediately have $BQ'/Q'C = 1$, making Q' the midpoint of BC. Then the line AQ is isogonaly conjugated to the median, implying the required statement.

    Since P belongs to the polars of B and C, then the points B and C belong to the polar of the point P, and we conclude that the polar of P is precisely BC. Consider the intersection D of the line BC with the tangent to the circumcircle at A. Since the point D belongs to the polars of A and P, AP has to be the polar of D. Hence $\mathscr{H}(B,C;D,Q)$. Let us now calculate the ratio BD/DC. Since the triangles ABD and CAD are similar we have $BD/AD = AD/CD = AB/AC$. This implies $BD/CD = (BD/AD)(AD/CD) = AB^2/AC^2$. The relation $\mathscr{H}(B,C;D,Q)$ implies $BQ/QC = BD/DC = AB^2/AC^2$, which proves the statement.

17. The statement follows from the Brianchon's theorem applied to $APBMCN$.

18. Applying the Brianchon's theorem to the hexagon $AMBCPD$ we get that the line MP contains the intersection of AB and CD. Analogously, applying the Brianchon's theorem to $ABNCDQ$ we get that NQ contains the same point.

19. The Brokard's theorem claims that the polar of $F = AD \cap BC$ is the line $f = EO$. Since the polar of the point on the circle is equal to the tangent at that point we know that $K = a \cap d$, where a and d are polars of the points A and D. Thus $k = AD$. Since $F \in AD = k$, we have $K \in f$ as well. Analogously we can prove that $L \in f$, hence the points E, O, K, L all belong to f.

20. Let $G = AD \cap BC$. Let k be the circumcircle of $ABCD$. Denote by k_1 and k_2 respectively the circumcircles of $\triangle ADF$ and $\triangle BCF$. Notice that AD is the radical axis of the circles k and k_1; BC the radical axis of k and k_2; and FH the radical axis of k_1 and k_2. According to the famous theorem these three radical axes intersect at one point G. In other words we have shown that the points F, G, H are colinear.

    Without loss of generality assume that F is between G and H (alternatively, we could use the oriented angles). Using the inscribed quadrilaterals $ADFH$ and $BCFH$, we get $\angle DHF = \angle DAF = \angle DAC$ and $\angle FHC = \angle FBC = \angle DBC$, hence $\angle DHC = \angle DHF + \angle FHC = \angle DAC + \angle DBC = 2\angle DAC = \angle DOC$. Thus the points D, C, H, and O lie on a circle. Similarly we prove that the points A, B, H, O lie on a circle.

    Denote by k_3 and k_4 respectively the circles circumscribed about the quadrilaterals $ABHO$ and $DCHO$. Notice that the line AB is the radical axis of the circles k and k_3. Simlarly CD and OH, respectively, are those of the pairs of circles (k, k_2) and (k_3, k_4). Thus these lines have to intersect at one point, and that has to be E. This proves that the points O, H, and E are colinear.

    According to the Brocard's theorem we have $FH \perp OE$, which according to $FH = GH$ and $OE = HE$ in turn implies that $GH \perp HE$, q.e.d.